A Theoretical Study on Differential Settlements of Structures

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DISASTER PREVENTION RESEARCH INSTITUTE

BULLETIN No. 14 MARCH, 1956

A THEORETICAL STUDY ON DIFFERENTIAL SETTLEMENTS OF STRUCTURES

BY

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Preface

There have been various discussions[1)~10)] on differential settlements of a structure standing on soft foundation. As yet, however, enough attention has not been paid to the effect on differential settlements of the rigidity of a structure which certainly is very effective thereupon. In other words, although the effect of this rigidity has heretofore been taken into consideration in connection with the elastic settlements due to load on surface of foundation, its effect has been rather neglected in the case of the settlements due to consolidation which assuredly occupy the major part of the differential settlements on soft foundation; except that Biot[11)] tried to solve this problem concerning the strip load. This essay contains several discussions on the settlements due to consolidation of clay stratum and on the settlements due to the elastic deformation of foundation, considering the rigidity of the structure in both cases. The essay is divided into Part A and Part B.

Part A is the discussion mainly on the case of the settlements due to consolidation. In this part, with the assumption that a structure deforms in proportion to shearing force, it is pointed out that the effect of the rigidity on differential settlements is very great. Some considerations are also given on elastic settlements.

Part B contains the discussion on final quantities of differential settlements which are necessary in the design of a structure, on the basis of Part A, and the method of calculation of final quantities of differential settlements. The discussion of derivation of the theories for Part B is shown in Chapter 10.

The assumptions adopted in this essay are as follows.

a) The Terzaghi's Consolidation Theory is available for the underground clay stratum.

b) Exclusively the one-dimensional and primary consolidation is under consideration, and the effects of the two-dimensional and secondary consolidations are not taken into consideration.

c) Whether there is permeable or impermeable stratum or not, the distribution of stress through foundation is displayed by the Boussinesq's formulas.

d) The structure is to be deformed completely elastically and its creep deformation is under no consideration.

Part A The settlement process of a structure due to the consolidation of underground clay stratum

§ 1 The consolidation equation in connection with pressure fluctuation

Primarily it is assumed that underground clay stratum infinitely extends horizontally between the two, upper and lower, permeable strata, and only the vertical or one-dimensional consolidation is under consideration. Further it is assumed that the pressure on clay stratum is constant through the thickeness and equals to the value at the centre of the thickness.

According to the assumptions mentioned above, the following settlement equation is obtained for the constant increase of pressure on clay stratum by means of the Terzaghi's Consolidation Theory.

$$y=\lambda(p-p_1)\left\{1-\sum_{m=0}^{\infty}\frac{2}{M^2}e^{-M^2T}\right\}, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

where $\lambda=ad/(1+e)$: final quantity of settlement at $p-p_1=1$,

p : excess hydrostatic pressure at t,

p_1 : initial value of p,

a : coefficient of compressibility,

d : thickness of clay stratum,

c : coefficient of consolidation,

t : time,

e : void ratio,

$$M=(2m+1)\pi/2, \quad T=4ct/d^2.$$

When $p-p_1=q$ is assumed, (1) approximately becomes

$$y=\lambda q(1-e^{-N}), \qquad \cdots\cdots\cdots\cdots\cdots\cdots(2)$$

where

$$N=\frac{\pi^2}{4}T.$$

When p is a function of N and accordingly q is its function as well, the following equation is obtained by means of the Duhamel's Theorem.

$$y=\lambda e^{-N}\int_0^N qe^{\nu}d\nu. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(3)$$

The case when several clay strata exist among permeable strata can be discussed in the like manner, though it is not stated here. (For the criticism on the treatment that approximately (2) instead of (1) is used in this chapter, see Appendix 1.)

§ 2 Differential equations concerning settlements of a structure

It is considered that there is such a structure as in Fig. 1 on the surface

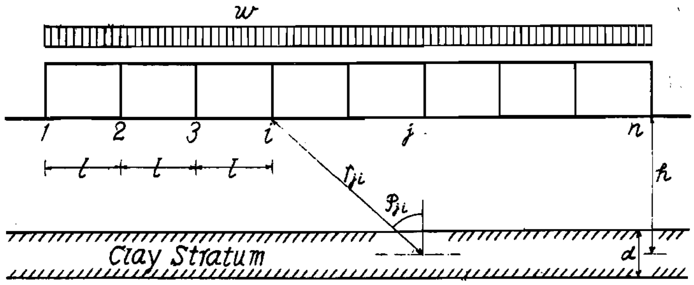

Fig. 1

of foundation and assumed to treat as a two-dimensional problem, the pressure of columns on a line perpendicular to the figure is approximated by a line load extending infinitely in the same direction.

If notations as follows are assumed under the column j,

α_{ji} : effect of reaction of base i on pressure on clay stratum under base j,

β_{ji} : effect of settlement of base i on reaction of base j,

δ_{ji} : effect of settlement of base i on pressure on clay stratum under base j,

γ_{ji} : effect of reaction of base i on elastic settlement of base j,

ρ_{ji} : effect of settlement of base i on elastic settlement of base j,

then, the following relations exist as to α, β, δ, γ and ρ,

$$\left.\begin{aligned} &\alpha_{ji}=\alpha_{ij},\\ &\beta_{ji}=\beta_{ij},\\ &\delta_{ji}=\sum_k \alpha_{jk}\beta_{ki},\\ &\gamma_{ji}=\gamma_{ij},\\ &\rho_{ji}=\sum_k \gamma_{jk}\beta_{ki}. \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(4)$$

When it is assumed that a structure has equal span lengths, the following expression is obtained.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13}\cdots a_{1n} \\ a_{21} & a_{22} & a_{23}\cdots a_{2n} \\ a_{31} & a_{32} & a_{33}\cdots a_{3n} \\ \vdots & \vdots & \vdots \quad \vdots \\ a_{n1} & a_{n2} & a_{n3}\cdots a_{nn} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots a_n \\ a_2 & a_1 & a_2 & \cdots a_{n-1} \\ a_3 & a_2 & a_1 & \cdots a_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots a_1 \end{pmatrix} \quad \cdots\cdots\cdots\cdots(5)$$

Furthermore, as for a_{ji}, when the foundation is taken as uniform semi-infinite elastic solid, the following equation is obtained by means of the formula of Boussinesq concerning line load in Fig. 1.

$$a_{ji} = \frac{2}{\pi}\frac{1}{r_{ji}}\cos^3\varphi_{ji} = \frac{2}{\pi}\frac{h^3}{r_{ji}^4}. \quad \cdots\cdots\cdots\cdots\cdots\cdots(6)$$

When base reaction is shown as

$$R_i = P_i + \sum_k \beta_{ik} y_k, \quad (P_i : \text{load of column } i), \cdots\cdots\cdots\cdots\cdots\cdots(7)$$

the increase of pressure on clay stratum is

$$q_i = \sum_j a_{ij} R_j = \sum_j a_{ij}(P_j + \sum_k \beta_{jk} y_k) = K_i + \sum_k \delta_{ik} y_k, \cdots\cdots\cdots\cdots\cdots(8)$$

where

$$K_i = \sum_j a_{ij} P_j.$$

When elastic settlement is considered in addition, the settlement equation of base i at $N=N$ is

$$y_i = \lambda e^{-N}\int_0^N q_i e^{\nu} d\nu + \sum_j \gamma_{ij} R_j. \quad \cdots\cdots\cdots\cdots\cdots\cdots(9)$$

When (7) and (8) are substituted into (9),

$$y_i = \lambda e^{-N}\int_0^N (K_i + \sum_k \delta_{ik} y_k) e^{\nu} d\nu + \sum_j \gamma_{ij}(P_j + \sum_k \beta_{jk} y_k), \quad \cdots\cdots\cdots(10)$$

therefore

$$\frac{dy_i}{dN} = \sum_k \rho_{ik}\frac{dy_k}{dN} + \sum_j \gamma_{ij}(P_j + \sum_k \beta_{jk} y_k) - y_i + \lambda(K_i + \sum_k \delta_{ik} y_k)$$

or

$$(\rho_{ii} - 1)\frac{dy_i}{dN} + \sum_k \rho_{ik}\frac{dy_k}{dN} + (\rho_{ii} + \lambda\delta_{ii} - 1) y_i$$
$$+ \sum_k(\rho_{ik} + \lambda\delta_{ik}) y_k + Q_i + \lambda_i K_i = 0 \quad (i \neq k), \quad \cdots(11)$$

where

$$Q_i = \sum_j \gamma_{ij} P_j.$$

(11) becomes at $N=0$

$$(1-\lambda\delta_{ii})y_i=\lambda(K_i+\sum_k \delta_{ik}y_k). \qquad \cdots\cdots(12)$$

§ 3 The solution when clay stratum is single

To solve the fundamental settlements' equations (11), those solutions are assumed as

$$\left.\begin{aligned} y_i &= \sum_m A_{im}(1-e^{-\Psi_m N})+B_i, \\ \frac{dy_i}{dN} &= \sum_m A_{im}\Psi_m e^{-\Psi_m N} \end{aligned}\right\} \qquad \cdots\cdots(13)$$

Substituting (13) into (11),

$$\begin{aligned} &[\sum_m\{(\rho_{ii}-1)(\Psi_m-1)-\lambda\delta_{ii}\}A_{im}+\sum_k\sum_m\{\rho_{ik}(\Psi_m-1)-\lambda\delta_{ik}\}A_{km}]e^{-\Psi_m N} \\ &\quad +(\rho_{ii}+\lambda\delta_{ii}-1)\{\sum_m A_{im}+B_i\}+\sum_k(\rho_{ik}+\lambda\delta_{ik})\{\sum_m A_{km}+B_k\} \\ &\quad\quad +Q_i+\lambda K_i=0. \end{aligned} \qquad \cdots\cdots(14)$$

There the following two conditions are obtained.

$$\{(\rho_{ii}-1)(\Psi_m-1)/\lambda-\delta_{ii}\}A_{im}+\sum_k\{\rho_{ik}(\Psi_m-1)/\lambda-\delta_{ik}\}A_{km}=0, \quad \cdots\cdots(15)$$

$$\begin{aligned} &(\rho_{ii}+\lambda\delta_{ii}-1)\{\sum_m A_{im}+B_i\} \\ &\quad +\sum_k(\rho_{ik}+\lambda\delta_{ik})\{\sum_m A_{km}+B_k\}+Q_i+\lambda K_i=0. \end{aligned} \qquad \cdots\cdots(16)$$

In order that (15) may be satisfied,

$$\begin{vmatrix} (\rho_{11}-1)x-\delta_{11} & \rho_{12}x-\delta_{12} & \cdots\cdots & \rho_{1n}x-\delta_{1n} \\ \rho_{21}x-\delta_{21} & (\rho_{22}-1)x-\delta_{22} & \cdots\cdots & \rho_{2n}x-\delta_{2n} \\ \vdots & \vdots & & \vdots \\ \rho_{n1}x-\delta_{n1} & \rho_{n2}-\delta_{n2} & \cdots\cdots & (\rho_{nn}-1)x-\delta_{nn} \end{vmatrix}=0, \qquad \cdots\cdots(17)$$

where

$$x=(\Psi_m-1)/\lambda.$$

From (17) Ψ_m are obtained as n roots, and the ratios of A_{ij} are determined from (15). From (16) $(\sum_m A_{im}+B_i)$ are determined.

And as the initial condition at $N=0$, from (13)

$$y_i=B_i. \qquad \cdots\cdots(18)$$

Therefore, (18) being substituted into (12),

$$(1-\lambda\delta_{ii})B_i=\lambda(K_i+\sum_k \delta_{ik}B_k), \qquad \cdots\cdots\cdots\cdots\cdots\cdots(19)$$

from which B_i are obtained. Thus $A_{im}=a_{im}A_m$ being set, from (16)

$$(\rho_{ii}+\lambda\delta_{ii}-1)\{\sum_m a_{im}A_m+B_i\}$$
$$+\sum_k(\rho_{ik}+\lambda\delta_{ik})\{\sum_m a_{km}A_m+B_k\}+Q_i+\lambda k_i=0, \quad \cdots\cdots\cdots\cdots\cdots(20)$$

where A_m are determined and solutions (13) are obtained.

Below, first the case concerning consolidation of clay stratum only, then the case concerning elastic deformation of foundation only are considered, and lastly both cases are considered together at the end of Chapter 9.

First, when consolidation of clay stratum only is considered, the equations above are simplified as follows,

from (11)

$$\frac{dy_i}{dN}+(1-\lambda\delta_{ii})y_i=\lambda(K_i+\sum_k \delta_{ik}y_k) \quad (i\neq k), \cdots\cdots\cdots\cdots\cdots\cdots(21)$$

from (13)

$$y_i=\sum_m A_{im}(1-e^{-\Psi_m N}) \qquad \begin{matrix} i=1,\ 2,\cdots\ n \\ m=1,\ 2,\cdots\ n, \end{matrix} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(22)$$

from (15) and (16)

$$\{(\Psi_m-1)/\lambda+\delta_{ii}\}A_{im}+\sum_k \delta_{ik}A_{km}=0 \qquad (i\neq k), \quad \cdots\cdots\cdots\cdots\cdots(23)$$

$$(\lambda\delta_{ii}-1)\sum_m A_{im}+\lambda\sum_k \delta_{ik}\sum_m A_{km}+\lambda K_i=0 \qquad (i\neq k), \quad \cdots\cdots\cdots\cdots(24)$$

from (17)

$$\begin{vmatrix} \delta_{11}+x & \delta_{12} & \cdots\cdots & \delta_{1n} \\ \delta_{21} & \delta_{22}+x & \cdots\cdots & \delta_{2n} \\ \vdots & \vdots & & \vdots \\ \delta_{n1} & \delta_{n2} & \cdots\cdots & \delta_{nn}+x \end{vmatrix}=0. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(25)$$

§ 4 The case in which the shearing rigidity of a structure is considered

It is assumed that a structure has uniform span lengths and uniform rigidity,

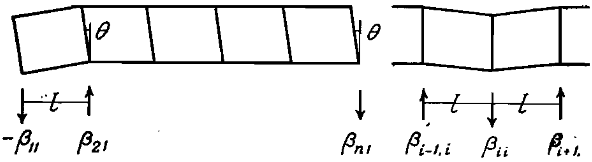

Fig. 2

and that, for the simplication, the plane of framework of each span is deformed in proportion with its shearing force. Then, with reference to Fig. 2, β_{ij} are

for the intermediate columns

$$\beta_{i-1,i}=\beta_{i+1,i}=\beta, \qquad \beta_{ii}=-2\beta, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(26)$$

for the end columns,

$$\left.\begin{aligned}&\beta_{11}\times 1=\beta_{n1}\times(n-2), && \beta_{11}+\beta_{n1}=\beta_{21},\\ &\beta(1/l-\theta)=\beta_{11}/l, && \beta\theta=\beta_{n1}/l,\end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(27)$$

accordingly β take the values as follows,

$$\left.\begin{aligned}&\beta_{11}=\beta_{nn}=-(n-2)\beta/(n-1),\\ &\beta_{1n}=\beta_{n1}=-\beta/(n-1),\\ &\beta_{n-1,n}=\beta_{21}=\beta.\end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(28)$$

Except the above mentioned (26) and (28), β_{ij} become zero.
Further when

$$\Delta a_i=a_{i+1}-a_i, \quad \Delta^2 a_i=\Delta a_i-\Delta a_{i-1}=a_{i+1}-2a_i+a_{i-1}, \quad \cdots\cdots\cdots\cdots(29)$$

where

$$\Delta^2 a_1=2\Delta a_1$$

are assumed, δ_{ij} are expressed by each row of the following matrix being multiplied by β,

$$\begin{pmatrix} \Delta a_1+(a_1-a_n)/(n-1) & \Delta^2 a_2 & \Delta^2 a_3 \cdots \Delta^2 a_{n-1} & -\Delta a_{n-1}-(a_1-a_n)/(n-1) \\ -\Delta a_1+(a_2-a_{n-1})/(n-1) & \Delta^2 a_1 & \Delta^2 a_2 \cdots \Delta^2 a_{n-2} & -\Delta a_{n-2}-(a_2-a_{n-1})/(n-1) \\ -\Delta a_2+(a_3-a_{n-2})/(n-1) & \Delta^2 a_2 & \Delta^2 a_1 \cdots \Delta^2 a_{n-3} & -\Delta a_{n-3}-(a_3-a_{n-2})/(n-1) \\ \vdots & \vdots & \vdots \quad\;\; \vdots & \vdots \\ -\Delta a_{n-1}+(a_n-a_1)/(n-1) & \Delta^2 a_{n-1} & \Delta^2 a_{n-2} \cdots \Delta^2 a_2 & +\Delta a_1-(a_n-a_1)/(n-1) \end{pmatrix}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots(30)$$

If a structure is under symmetrical loads the matrix of the coefficients of A_{ij} of (23) becomes, in the case of $n=2m+1$,

$$\begin{pmatrix} (\Delta a_1-\Delta a_{n-1})+x & \Delta^2 a_2+\Delta^2 a_{n-1} & \Delta^2 a_3+\Delta^2 a_{n-2}\cdots\Delta^2 a_m+\Delta^2 a_{m+2} & \Delta^2 a_{m+1} \\ -(\Delta a_1+\Delta a_{n-2}) & (\Delta^2 a_1+\Delta^2 a_{n-2})+x & \Delta^2 a_2+\Delta^2 a_{n-3}\cdots\Delta^2 a_{m-1}+\Delta^2 a_{m+1} & \Delta^2 a_m \\ \vdots & \vdots & \vdots \qquad\qquad \vdots & \vdots \\ -(\Delta a_{m-1}+\Delta a_{m+1}) & \Delta^2 a_{m-1}+\Delta^2 a_{m+1} & \Delta^2 a_{m-2}+\Delta^2 a_{m+1}\cdots(\Delta^2 a_1+\Delta^2 a_3)+x & \Delta^2 a_2 \\ -(\Delta a_m+\Delta a_m) & \Delta^2 a_m+\Delta^2 a_m & \Delta^2 a_{m-1}+\Delta^2 a_{m-1}\cdots\Delta^2 a_2+\Delta^2 a_2 & \Delta^2 a_1+x \end{pmatrix}, \qquad \cdots\cdots(31)$$

where $x=(\Psi_j-1)/\beta\lambda$. The case at $n=2m$ also is solved in the same way. The equation corresponding to (25) is obtained by putting the determinant of (31) into zero.

§ 5 Considerations on solutions in the case of settlements due to consolidation

If the solutions concerning $(2n+1)$-span and $2n$-span symmetrical rigid frames under uniform distribution of load and with uniform rigidity are obtained on the basis of the above mentioned theoretical formulas, it is experessed as follows.

$$\left.\begin{aligned} y_1 &= f(\xi)\frac{P_1\lambda}{l}(1-e^{-N})+g_{11}(\xi,\eta)\frac{P_1\lambda}{l}(1-e^{-\Psi_1 N})+ \\ &\qquad \cdots\cdots+g_{1,n-1}(\xi,\eta)\frac{P_1\lambda}{l}(1-e^{-\Psi_{n-1}N}) \\ y_2 &= f(\xi)\frac{P_1\lambda}{l}(1-e^{-N})+g_{21}(\xi,\eta)\frac{P_1\lambda}{l}(1-e^{-\Psi_1 N})+ \\ &\qquad \cdots\cdots+g_{2,n-1}(\xi,\eta)\frac{P_1\lambda}{l}(1-e^{-\Psi_{n-1}N}) \\ &\vdots \\ y_n &= f(\xi)\frac{P_1\lambda}{l}(1-e^{-N})+g_{n1}(\xi,\eta)\frac{P_1\lambda}{l}(1-e^{-\Psi_1 N})+ \\ &\qquad \cdots\cdots+g_{n,n-1}(\xi,\eta)\frac{P_1\lambda}{l}(1-e^{-\Psi_{n-1}N}) \end{aligned}\right\} \cdots\cdots\cdots(32)$$

Therefore the differential settlements, the differences between settlements of neighbouring couple of bases, are

$$y_{i+1\sim i}=y_{i+1}\sim y_i=\sum_{m=1}^{n-1}\{g_{i+1,m}(\xi,\eta)\sim g_{i,m}(\xi,\eta)\}\frac{P_1\lambda}{l}(1-e^{-\Psi_m N}), \cdots\cdots(33)$$

where $\quad \xi=l/h,\quad \eta=\beta\lambda/l,\quad \Psi_m=1+Q_m(\xi,\eta)$ and $P_1=wl/2$.

For example, (32) and (33) become in the case of 3 spans

$$\left.\begin{aligned}
y_1=y_4&=\frac{(\Delta a_1'+\Delta a_2')\bar{K}_1+(\Delta a_1'-\Delta a_3')\bar{K}_2}{(2\Delta a_1'+\Delta a_2'-\Delta a_3')}\frac{P_1\lambda}{l}(1-e^{-N})\\
&\quad+\frac{(\bar{K}_2-\bar{K}_1)(\Delta a_3'-\Delta a_1')}{\Psi(2\Delta a_1'+\Delta a_2'-\Delta a_3')}\frac{P_1\lambda}{l}(1-e^{-\Psi N})\\
y_2=y_3&=\frac{(\Delta a_1'+\Delta a_2')\bar{K}_1+(\Delta a_1'-\Delta a_3')\bar{K}_2}{(2\Delta a_1'+\Delta a_2'-\Delta a_3')}\frac{P_1\lambda}{l}(1-e^{-N})\\
&\quad+\frac{(\bar{K}_2-\bar{K}_1)(\Delta a_1'+\Delta a_2')}{\Psi(2\Delta a_1'+\Delta a_2'-\Delta a_3')}\frac{P_1\lambda}{l}(1-e^{-\Psi N})
\end{aligned}\right\} \quad\cdots\cdots\cdots(34)$$

and

$$y_{2-1}=y_2-y_1=\frac{\bar{K}_2-\bar{K}_1}{\Psi}\frac{P_1\lambda}{l}(1-e^{-\Psi N}), \qquad\cdots\cdots\cdots\cdots\cdots\cdots(35)$$

where
$$\Delta a_i'=\Delta a_i l=(a_{i+1}-a_i)\,l=a_{i+1}'-a_i',$$
$$\bar{K}_1=a_1'+2a_2'+2a_3'+a_4',\quad \bar{K}_2=2a_1'+3a_2'+a_3',$$
$$\Psi=1-(2\Delta a_1'+\Delta a_2'-\Delta a_3')\,\eta.$$

If the units of various coefficients are given as a reference, β, λ, P_1 and $\frac{P_1\lambda}{l}$ are kg/cm^2, cm^3/kg, kg/cm, and cm respectively, and a', ξ, and η are zero dimension.

The characters of (32) and (33) are explained below mainly through (34) and (35).

a) As shown in (32) and (33), the settlement of each base can be divided

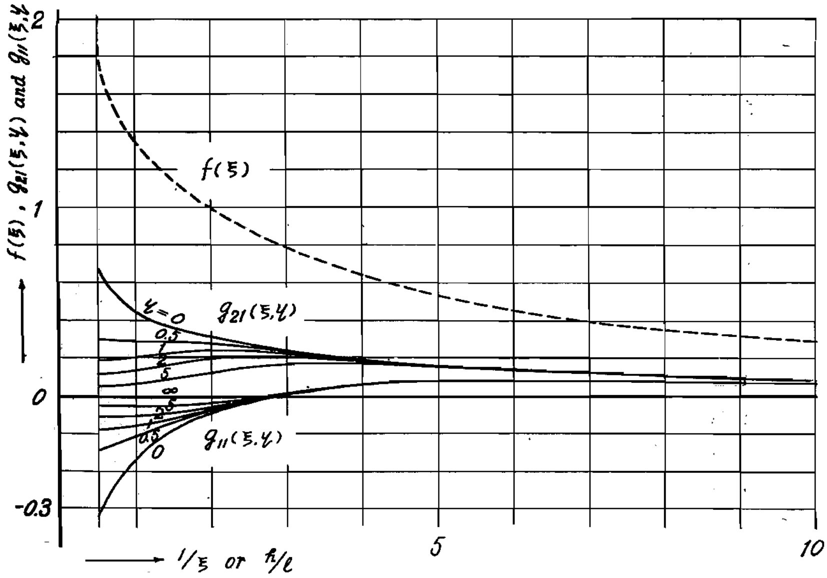

Fig. 3 Final quantities of uniform (A-type) settlement and non-uniform (B-type) settlement due to consolidation in the case of a 3-span symmetrical rigid frame.

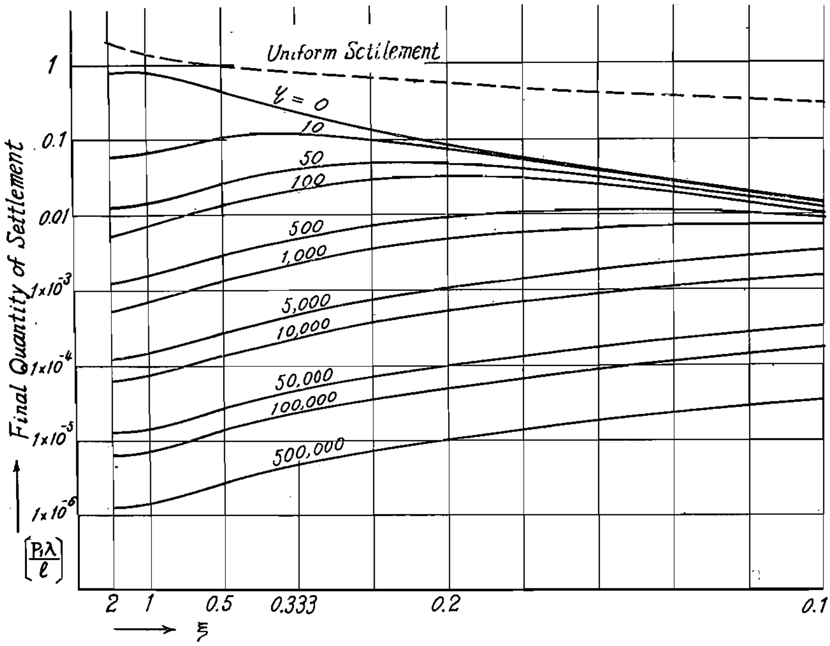

Fig. 4 Final qnantities of uniform (A-type) and differential settlements due to consolidation when η is considerably great in the case of a 3-span symmetrical rigid frame.

into two parts, i.e. term of uniform settlement $f(\xi)\frac{P_1\lambda}{l}(1-e^{-N})$ (denoted as settlement type A) and terms of non-uniform settlements $\sum g_{lm}\frac{P_1\lambda}{l}(1-e^{-\Psi_m N})$ (denoted as settlement type B). Here the coefficient of term of settlement type A is inversely proportional to the depth of clay stratum from the surface of foundation and independent of the rigidity of the structure. The greater rigidity β is, the more rapidly the coefficient g_{lm} of each term of settlement type B decreases, and it converges into zero when $\gamma\ (=\beta\lambda/l)\rightarrow\infty$. The effect of rigidity β intends to decrease with the increasing of the depth of the location of the clay stratum. In the case of 3 spans in (34) and (35), $f(\xi)$, $g_{11}(\xi,\gamma)$ and $g_{21}(\xi,\gamma)$ are shown in Fig. 3. And in Fig. 4 are shown $f(\xi)$ and final quantity of differential settlement $\{g_{21}(\xi,\gamma)-g_{11}(\xi,\gamma)\}$ in case γ is great.

b) At constant ξ and γ, settlement of each base approaches to the maximum with $N\rightarrow\infty$. The greater β is, the greater the settlement function $(1-e^{-\Psi_l N})$ of each term type B is, and it coincides with function of settlement type A $(1-e^{-N})$ at $\beta=0$, and it intends to approach asymptotically to settlement function of type

A for smaller ξ. Examples of curve $(1-e^{-\Psi N})\sim N$ in the case of 3 spans are shown in Fig. 5.

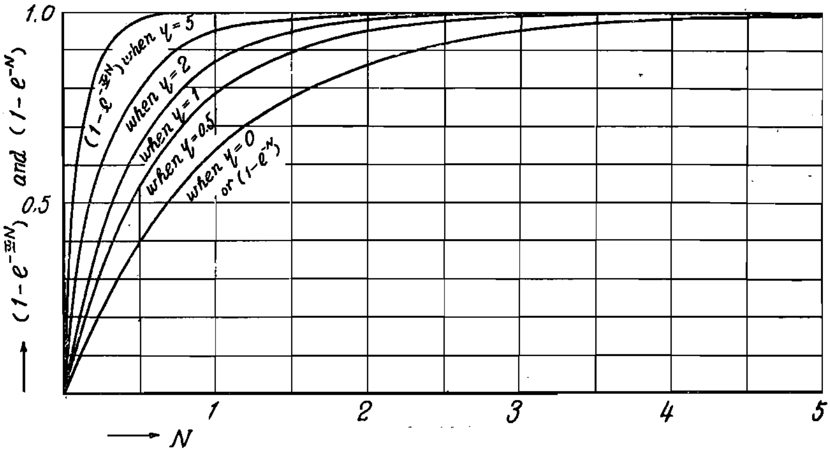

Fig. 5 Function of settlement due to consolidation, $(1-e^{-\Psi N})\sim N$ curves when η varies in the case of a 3-span symmetrical rigid frame.

c) An example of final quantities of settlements, in case the number of span increases, is shown in Fig. 6. In the case of infinite number of spans, in differential equation (21)

$$\left.\begin{aligned} y_i &= y_k = y, \\ K_i &= K = \sum_{j=-\infty}^{\infty} a_{ij} P_j, \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots(36)$$

therefore, as $\sum_k \delta_{ik}=0$, the following equation is obtained

$$y=\lambda K(1-e^{-N}). \qquad \cdots\cdots\cdots\cdots(37)$$

In Fig. 6, it is known that final quantity of maximum settlement in the case of 7 spans is 0.84 times that in the case of infinite number of spans, and that with the increasing of number of spans settlement of each base converges considerably fast.

d) Further in case η is great enough, clay constant comes to be not remarkably effective on differential settlements. For example, in the case of 3 spans, as shown in (34), η of $g_{11}(\xi,\eta)$ and $g_{21}(\xi,\eta)$ are included in Ψ of denominators, but in the case of $(2\Delta a_1'+\Delta a_2'-\Delta a_3')\eta \gg 1$, 1 can be neglected. In general, in the case of the number of spans $2n+1$ and $2n$, $g_{im}(\xi,\eta)$ becomes the equation in which denominator is of n-th degree of η and numerator is of

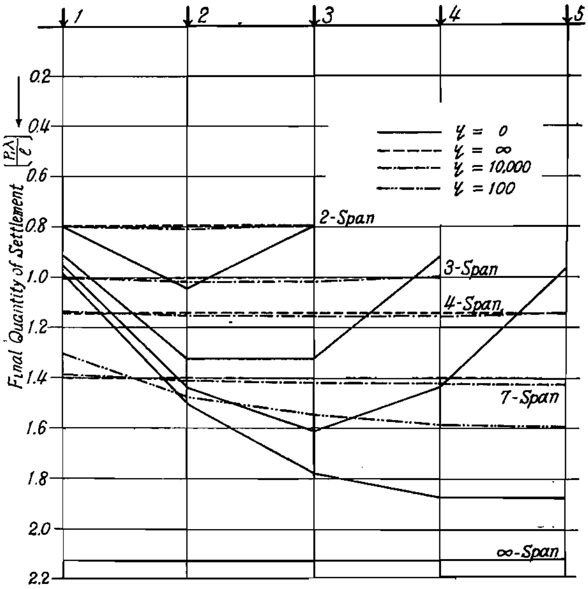

Fig. 6 Final quantities of settlements of bases due to consolidation when the number of columns varies. (ξ=0.5)

$(n-1)$-th degree of γ, and if others are neglected for the term with the greatest power of γ, each term of type B, i.d. final quantities of differential settlements can be regarded to be independent of λ, for λ is cancelled out in denominator and numerator. Thus it is pointed out that with the increasing of β differential settlements come to be little dependent of the property of clay.

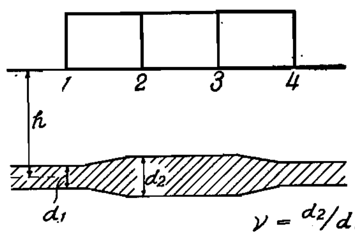

Fig. 7 Clay stratum with section in the form of lens.

e) When the thickness of clay stratum changes under each base and accordingly λ changes, if section of clay stratum in the form of lens as shown in Fig. 7 is assumed, for the case of 3 spans

$$\frac{f(\xi)_{\nu=\nu}}{f(\xi)_{\nu=1}}=\frac{(\varDelta a_1'+\varDelta a_2')+(\varDelta a_1'-\varDelta a_3')}{\nu(\varDelta a_1'+\varDelta a_2')+(\varDelta a_1'-\varDelta a_3')}\nu\ , \qquad \cdots\cdots(38)$$

$$\frac{y_{2\sim1,\nu=\nu}}{y_{2\sim1,\nu=1}}=\frac{\bar{K}_1-\nu\bar{K}_2}{\bar{K}_1-\bar{K}_2}\,\frac{\{(\varDelta a_1'+\varDelta a_2')+(\varDelta a_1'-\varDelta a_3')\}\gamma-1}{\{(\varDelta a_1'+\varDelta a_2')\nu+(\varDelta a_1'-\varDelta a_3')\}\gamma-1} \qquad \cdots\cdots(39)$$

Fig. 8 shows the case in which γ becomes great enough and 1 in denominator

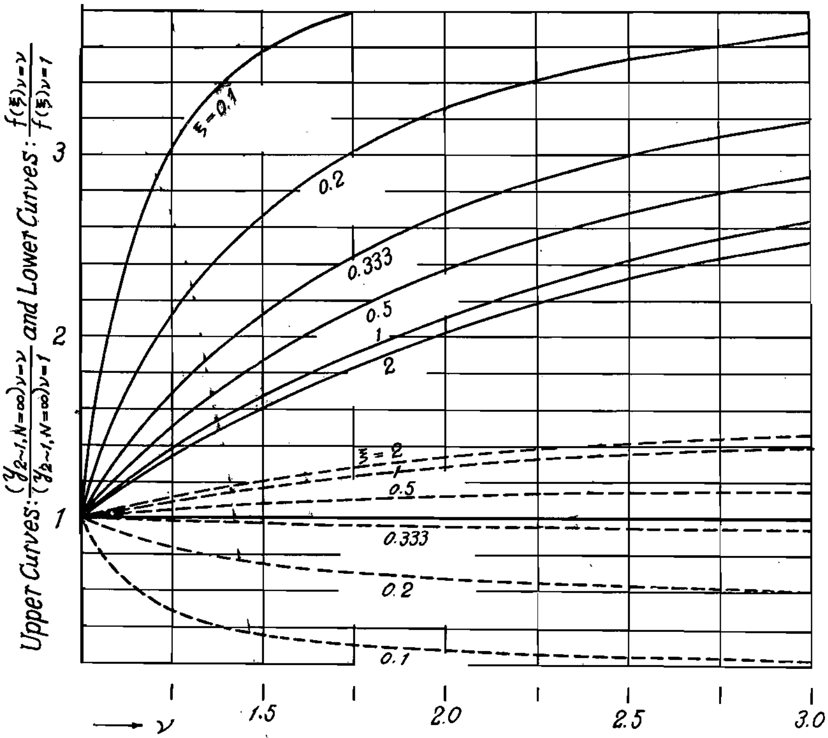

Fig. 8 The effect of ν (ratio of thickness in Fig. 7) on final quantities of A-type and differential settlements due to consolidation in the case of a 3-span symmetrical rigid frame. (when $(2\Delta\alpha_1'+\Delta\alpha_2'-\Delta\alpha_3')\eta\gg1$ in Eq. (39))

and numerator can be neglected in (38) and (39). From this figure it is clearly known that the slight change of ν is greatly effective on differential settlements.

f) when structures are extended

Although only the cases of symmetrical settlements are treated above, the case of extended structures in which such damages of structure as cracks and plastic hinges of members of structure are apt to occur belongs to the case of asymmetrical load situation. The column numbers of an original structure are denoted as 1, 2, ⋯⋯m, and that of its extended part as $m+1$, $m+2$, ⋯⋯n. If it is assumed that the original structure is through with the settlement phenomenon by the consolidation of clay stratum, the settlement equation of each column when the structure is extended is

$$\frac{dy_i}{dN}+y_i=\lambda_i(K_i+\sum_k\delta_{ik}y_k) \qquad \begin{matrix} i=1,2,\cdots\cdots n \\ k=1,2,\cdots\cdots n, \end{matrix} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(40)$$

where $$K_i=\sum_j a_{ij}P_j \ ,$$

$$P_1 = P_2 = \cdots\cdots\cdots = P_{m-1} = 0, \qquad P_m = P_n = wl/2 \ ,$$

hence equation (40) is solved as in Chapter 3. The differences of λ_i among i which come from the consolidation by loads of the original structure are usually so little as to be neglected, so λ_i can be let as λ in (40).

§ 6 When the rigidity of a structure changes

So far, from Chapter 2 to Chapter 5, rigidity β has been let as constant from $N=0$. As the result, it has been concluded that, as later shown in Chapter 9, there can be cases in which the major part of differential settlement takes place as early as in the first few months. This shows that the effect of variation of rigidity with the hardening of concrete must be thought in the case of a reinforced concrete structure. Therefore the further consideration comes from the assumption that rigidity changes in the way of approaching to constant value in the interval of infinite time.

The rigidity of a structure is assumed as $\beta\varphi(N)$. Here

$$\varphi(0) = 0 \ , \qquad \varphi(N)_{N\to\infty} \to 1 \qquad \cdots\cdots\cdots\cdots\cdots\cdots(41)$$

If the constancy of load is assumed from $N=0$,
(21) are

$$\frac{dy_i}{dN} + y_i = \lambda\{K_i + \sum_k \delta_{ik}\varphi(N) y_k\} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(42)$$

$$i = 1,2,\cdots\cdots\cdots n,$$
$$k = 1,2,\cdots\cdots\cdots n.$$

Those solutions are put as follows,

$$y_i = \sum_j A_{ij} e^{-\int_0^N \Psi_j' dN} \int_0^N e^{\int_0^N \Psi_j' dN} dN \ . \qquad \cdots\cdots\cdots\cdots\cdots\cdots(43)$$

(43) being substituted into (42), the following two conditions are obtained from the coefficient terms of $e^{-\int_0^N \Psi_j' dN} \int_0^N e^{\int_0^N \Psi_j' dN} dN$ and constant terms,

$$A_{ij}(1 - \Psi_j') - \lambda \sum_k \delta_{ik}\varphi(N) A_{kj} = 0, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(44)$$

$$\sum A_{ij} - \lambda K_i = 0 \qquad \cdots\cdots\cdots\cdots\cdots\cdots(45)$$

In order that (44) may be satisfied,

$$\begin{vmatrix} \delta_{11}\varphi(N)+x & \delta_{12}\varphi(N) & \cdots\cdots\cdots\cdots \delta_{1n}\varphi(N) \\ \delta_{21}\varphi(N) & \delta_{22}\varphi(N)+x & \cdots\cdots\cdots\cdots \delta_{2n}\varphi(N) \\ \vdots & \vdots & \vdots \\ \delta_{n1}\varphi(N) & \delta_{n2}\varphi(N) & \cdots\cdots\cdots\cdots \delta_{nn}\varphi(N)+x \end{vmatrix} = 0, \quad \cdots\cdots\cdots\cdots\cdots\cdots (46)$$

where

$$x=(\Psi_j'-1)/\lambda\ .$$

Therefore, Ψ_i' are obtained from (46), and A_{ij} can be determined from (44) and (45), and accordingly (43) can be determined.

§ 7 Solution and consideration in the case of gradual increase of rigidity

If the case of 3-span structure (uniform span length, uniform rigidity, and uniform distributed load) is solved, the following equations are obtained,

$$\left.\begin{aligned} y_1=y_4&=\frac{(\varDelta\alpha_1'+\varDelta\alpha_2')\bar{K}_1+(\varDelta\alpha_1'-\varDelta\alpha_3')\bar{K}_2}{2\varDelta\alpha_1'+\varDelta\alpha_2'-\varDelta\alpha_3'}\frac{P_1\lambda}{l}(1-e^{-N}) \\ &+\frac{(\bar{K}_2-\bar{K}_1)(\varDelta\alpha_3'-\varDelta\alpha_1')}{2\varDelta\alpha_1'+\varDelta\alpha_2'-\varDelta\alpha_3'}\frac{P_1\lambda}{l}e^{-\int_0^N\Psi'dN}\int_0^N e^{\int_0^N\Psi'dN}dN, \\ y_2=y_3&=\frac{(\varDelta\alpha_1'+\varDelta\alpha_2')\bar{K}_1+(\varDelta\alpha_1'-\varDelta\alpha_3')\bar{K}_2}{2\varDelta\alpha_1'+\varDelta\alpha_2'-\varDelta\alpha_3'}\frac{P_1\lambda}{l}(1-e^{-N}) \\ &+\frac{(\bar{K}_2-\bar{K}_1)(\varDelta\alpha_1'+\varDelta\alpha_2')}{2\varDelta\alpha_1'+\varDelta\alpha_2'-\varDelta\alpha_3'}\frac{P_1\lambda}{l}e^{-\int_0^N\Psi'dN}\int_0^N e^{\int_0^N\Psi'dN}dN, \end{aligned}\right\} \cdots\cdots\cdots\cdots (47)$$

where $\quad \Psi'=1-(2\varDelta\alpha_1'+\varDelta\alpha_2'-\varDelta\alpha_3')\eta\varphi(N).$

$\bar{K}_1$ and $\bar{K}_2$ are the same that are in (34).

In comparison with solutions (34) in the case of rigidity $\beta=$constant, it is clearly known that solutions (47) also are divided into two parts of uniform settlement (type A) and non-uniform settlement (type B), and that the terms of uniform settlement are both alike and independent of rigidity. The coefficients of type B terms are completely the same except Ψ in the denominator of (34), and there is the ratio of $e^{-\int_0^N\Psi'dN}\int_0^N e^{\int_0^N\Psi'dN}dN\ :\ (1-e^{-\Psi N})/\Psi$ between the differential settlements $y_{2\sim1}$ in both cases. In the case of $\varphi(N)=1$, (47) completely coincides with (34).

§ 8 The case of elastic settlement

Although only the case of settlements due to consolidation has been considered above, the case of exclusively elastic settlement comes to be expressed by equation (12) as the terms concerning N disappear. Further, for the sake of simplification, as for γ, only the effect just under load is considered and the effects to the other bases are disregarded.

In the case of a symmetrical structure of 3 spans,

$$y_1=\left(\gamma+\frac{\beta\gamma^2}{2\beta\gamma+1}\right)P_1\ , \qquad \cdots\cdots\cdots(48)$$

$$y_2=\left(\gamma+\frac{\gamma^2\beta+\gamma}{2\beta\gamma+1}\right)P_1\ , \qquad \cdots\cdots\cdots(49)$$

and

$$y_{2\sim1}=\frac{\gamma}{2\beta\gamma+1}P_1\ . \qquad \cdots\cdots\cdots(50)$$

This case is also divided into uniform settlement (type A) of γP_1 and non-uniform settlement (type B). Settlement type A is independent of β and differential settlement rapidly decreases with the increasing of β.

§ 9 Numerical example 1.

B rigid frame of the example in the Instructions for Calculation of Reinforced Concrete Structures[12] written by the Architectural Institute of Japan is adopted as an example of the preceding calculation, as is shown in Fig. 9,

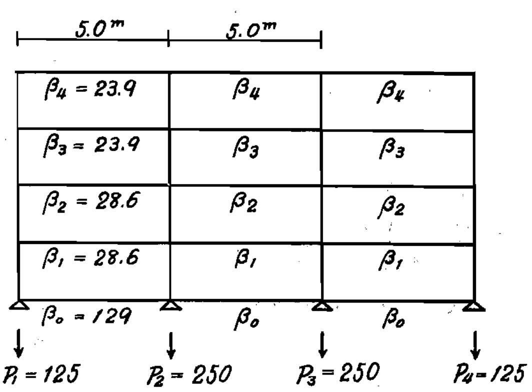

Fig. 9 The 3-span, 4-story rigid frame adopted in Capter 9.

structure : 4-story, 3-span, reinforced concrete rigid frame

$$\left.\begin{array}{ll} \text{rigidity of base beam :} & \beta_0=129 \text{ kg/cm}^2, \\ \text{rigidity of 1st floor beam :} & \beta_1=28.6 \text{ kg/cm}^2, \\ \text{total rigidity of beams :} & \beta=\sum_{s=0}^{4}\beta_s=234 \text{ kg/cm}^2, \\ P_1=125 \text{ kg/cm} , \quad l=500 \text{ cm} & \end{array}\right\} \quad \cdots\cdots\cdots(51)$$

For e, a and k, the following numerical values are taken,

$$\left.\begin{array}{l} e=1.1 , \quad a=0.07 \text{ cm}^2/\text{kg} , \\ k=5\times10^{-8} \text{ cm/sec.} \end{array}\right\} \quad \cdots\cdots\cdots(52)$$

Then

$$\frac{P_1\lambda}{l}=0.00835d , \qquad \eta=\frac{\beta\lambda}{l}=0.01563d. \quad \cdots\cdots\cdots(53)$$

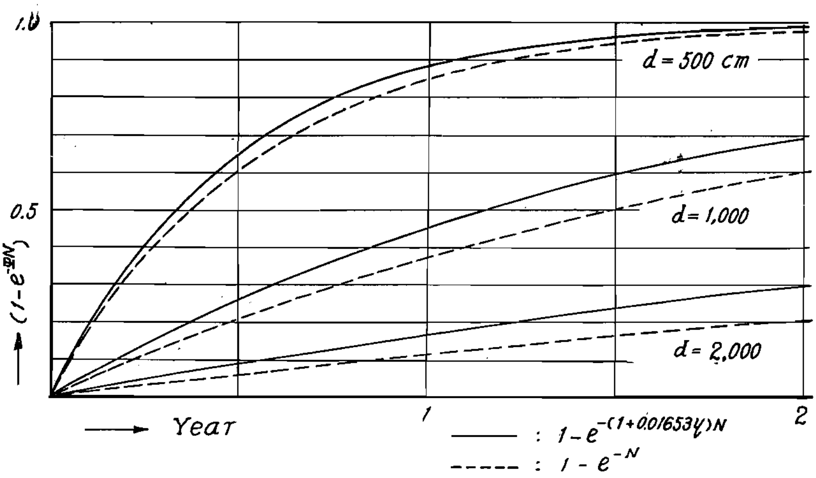

Fig. 10 Function of settlement due to consolidation, $(1-e^{-\Psi N})\sim N$ curves when d varies in the case of a 3-span symmetrical rigid frame. (β=234, ξ=0.2 and P_1= 125)

If the functions $(1-e^{-N})$ and $(1-e^{-\Psi N})$ of uniform settlement and differential settlement of 3-span symmetrical structure in Fig. 9 are expressed by these numerical values above, Fig. 10 and Fig. 11 are obtained. The case when the thickness of clay stratum d varies is shown in Fig. 10, and the case when ξ varies is shown in Fig. 11. It is clearly known from these two figures that the value of the function of differential settlement $y_{2\sim1}$ is somewhat greater than that of the function of uniform settlement, and that the deeper the position of clay stratum is, the smaller the ratio between the two intends to

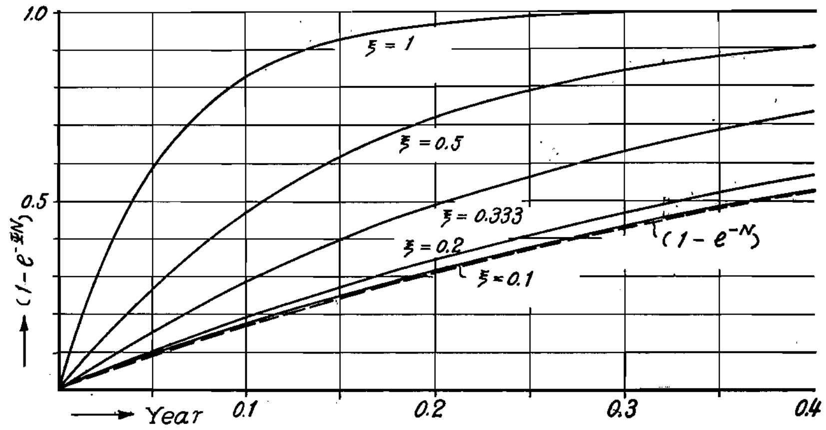

Fig. 11 Function of settlement due to consolidation, $(1-e^{-\Psi N})\sim N$ curves when ξ varies in the case of a 3-span symmetrical rigid frame. ($\beta=234$, $P_1=125$ and $d=500$)

be. An example of final quantities of uniform settlement and differential settlement is shown in Fig. 12. A numerical example of the increments of fibre stresses at ends of beams caused by final quantity of differential settlement is, for $\xi=0.2$ and $d=500\sim1{,}000$ cm,

$$\left.\begin{aligned} &f(\xi)\frac{P_1\lambda}{l}=4.18\sim8.35\text{ cm}\quad(3.50\sim7.00\text{ cm}),\\ &y_{2\sim1,N\to\infty}=0.302\sim0.541\text{ cm}\quad(0.900\sim1.545\text{ cm}),\\ &\Delta\sigma_1=38.3\sim68.7\text{ kg/cm}^2\quad(114\sim196\text{ kg/cm}^2),\\ &\Delta\sigma_0=80.2\sim144\text{ kg/cm}^2\quad(239\sim411\text{ kg/cm}^2).\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots(54)$$

where $\Delta\sigma_1$ and $\Delta\sigma_2$ show the increments of fibre stresses at ends of the first floor beam and the base beam respectively, and, as for the calculation, the reinforcement are neglected and the beams are assumed to have rectangular full sections of concrete. The parentheses show the case ($\nu=1.25$) when clay stratum has the section in the form of lens as in Fig. 7.

The increments of stresses shown in the above example are considerably large in comparison with allowable stress of concrete. In the case when there exists such a clay stratum as above, in a design of structure, by the effective Design Code, which disregards differential settlements, joints of frame are nearly in the state of plastic hinge and this fact is considered to prove the remarkable deformation of structures on soft foundation. It is also known from the above numerical example that the slight change of the thickness of clay stra-

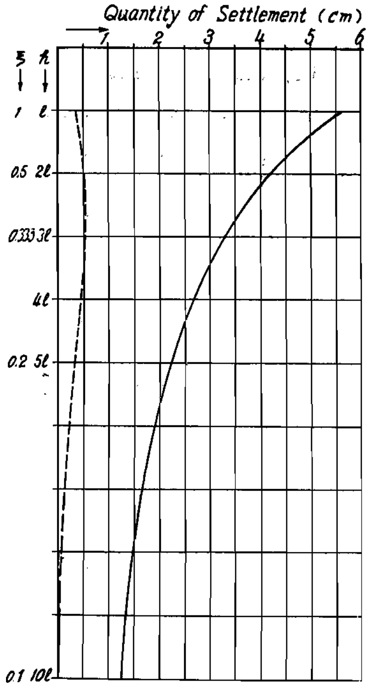

Fig. 12 Final quantities of uniform settlement (: the real curve) and differential settlement (: the dashed curve) due to consolidation when ξ varies in the case of a 3-span rigid frame. (β=234, P_1=125 and d=500)

tum has great influence on differential settlements.

As a numerical example for the case of the extension of a structure, a 4-span structure extended from 2-span structure is shown in Fig. 13. There are taken $\xi=0.2$, $d=500$ cm. It is clearly known from this figure that in this case the structure makes settlement type B as well as rotation and uniform settlement as a whole.. When 2-span or 4-span structures under uniform distribution of load are set from the beginning the final quantities of differential settlements are different from the differential settlements in the case mentioned above, and the differences between them are shown in the upper part of the figure for the comparison. The maximum slope occurs at the span between column 2 and 3, that means in case of extension the adjacent span to the original structure is apt to show

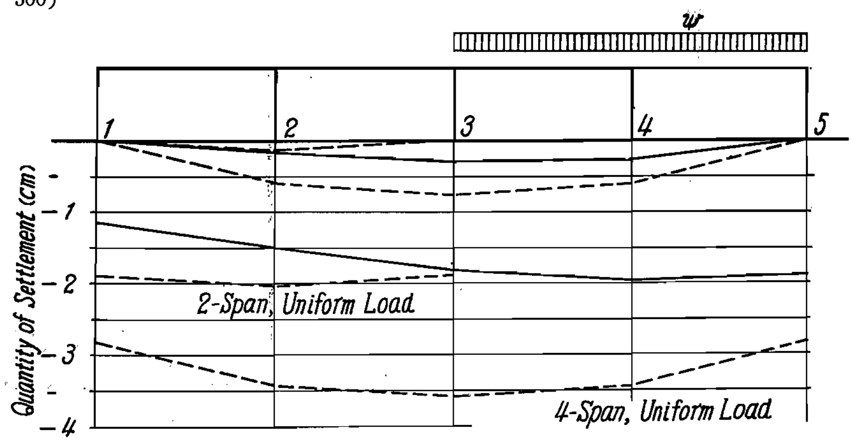

Fig. 13 Final quantities of settlements of bases due to consolidation in the case of a 4-span rigid frame extended from 2-span rigid frame, and those in the cases of a 2-span and 4-span rigid frames under uniform load which are not extended. (β=234, ξ=0.2, P_1=125 and d=500)

damages.

As an example for the case of varying rigidity due to the hardening of concrete, $\varphi(N)$ is assumed to be the curve of strength of concrete with respect to time, and the formula of Prof. S. Ban[13] is approximated to be

$$\varphi(t)=\frac{0.7\,t}{1.203+0.7\,t} \quad (t:\ \text{week}), \qquad \cdots\cdots\cdots\cdots\cdots\cdots(55)$$

If only the first story of the above 3-span, 4-story structure is assumed to have been built and $\xi=1$ and $d=500$ cm are taken, (55) becomes

$$\varphi(N)=\frac{N}{0.06231+N}, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(56)$$

where $\qquad N=0.03588\,t,$

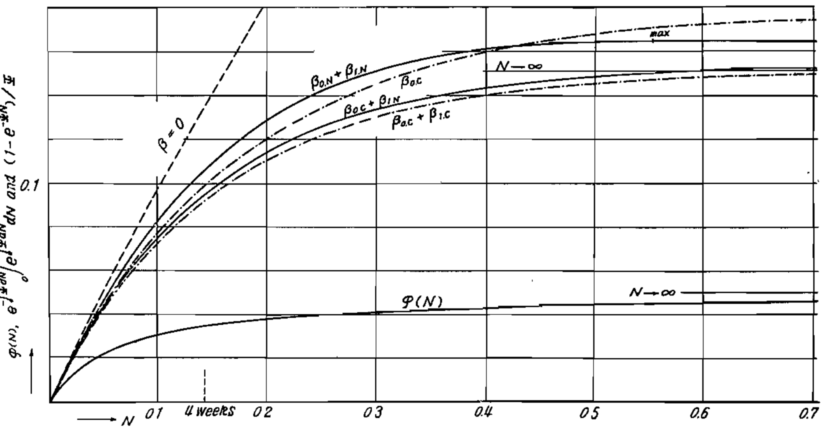

Fig. 14 The function of settlement due to consolidation. $e^{-\int_0^N \Psi' dN}\int_0^N e^{\int_0^N \Psi' dN}\, dN \sim N$ curves when the rigidity of structure inceases as $\beta\varphi(N)$ and the function of settlement $(1-e^{-\Psi N})/\Psi \sim N$ curves when the rigidity β_0=const. in the case of 4-span symmetrical rigid frame. ($\beta_0=129$, $\beta_1=28.6$, $\xi=1$ and $d=500$)

and Fig. 14 is obtained. Here suffix C shows the case in which rigidity β is constant, and suffix N shows the case in which rigidity changes with $\beta\varphi(N)$. In the same figure, the cases of $\beta=0$ and $\beta_{0C}+\beta_{1C}$ are also shown. It is known from this result that the cases of $\beta_{0N}+\beta_{1N}$ and $\beta_{0C}+_{1N}$ have greater gradient than that of $\beta_{0C}+\beta_{1C}$ at the beginning, and the formers reach the maximum values and gradually decrease, approaching to the final value

$$\lim_{N\to\infty}\frac{\int_0^N e^{\int_0^N \Psi' dN} dN}{e^{\int_0^N \Psi' dN}}=\lim_{N\to\infty}\frac{e^{\int_0^N \Psi' dN}}{\Psi' e^{\int_0^N \Psi' dN}}=\frac{1}{\Psi} \qquad \cdots\cdots\cdots\cdots(57)$$

It will be natural that the gradient of the curve in the case of $\beta_{0\sigma}+\beta_{1N}$ is between the gradient of the curve in the case of $\beta_{0\sigma}+\beta_{1\sigma}$ and that in the case of $\beta_{0\sigma}$. It is pointed out from the same figure that the maximum value in the case of $\beta_{0N}+\beta_{1N}$ shows about 110 % of the limit value and the variation of rigidity gives a dangerous effect in differential settlements. It is also known that the case of $\beta_{0\sigma}+\beta_{1N}$ is less different than the case of $\beta_{0N}+\beta_{1N}$ from the case of $\beta_{0\sigma}+\beta_{1\sigma}$, and, at the beginning of construction process of a structure, to set a base beam is considerably effective in decreasing differential settlements.

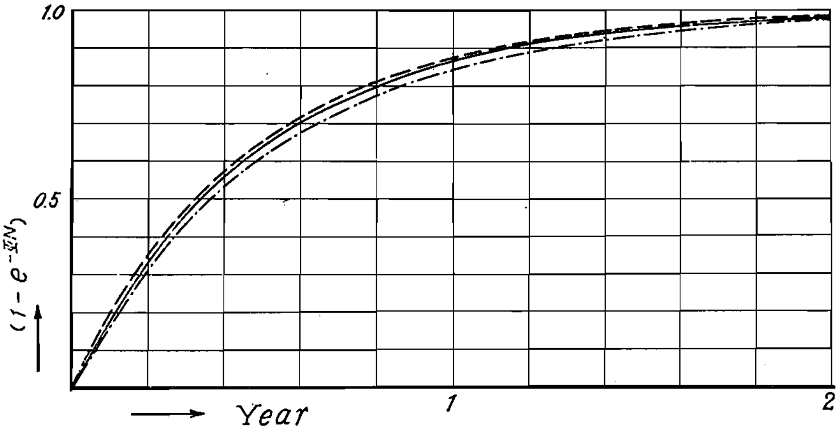

Fig. 15 The comparison among functions of uniform settlement (: the dash-dotted curve) and differential settlement (: the real curve) due to consolidation and elastic deformation of foundation both and that (: the dashed curve) due to consolidation alone. (β=234, P_1=125, d=500, ξ=0.2 and κ=6)

As a numerical example of the case which has elastic settlements as well, if $\xi=0.2$ and $d=500$ cm are taken, the result of the calculation is shown in Fig. 15 and Fig. 16.

Here the value of γ is approximately

$$\left.\begin{aligned} &\gamma=\frac{\delta}{P_1}=\frac{l'}{\kappa A},\\ \text{where}\quad &\kappa : \text{coefficient of subgrade reaction,}\\ &A : \text{area of base plate.}\end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots(58)$$

The coefficient γ will be expressed in Chapter 13 more in detail.

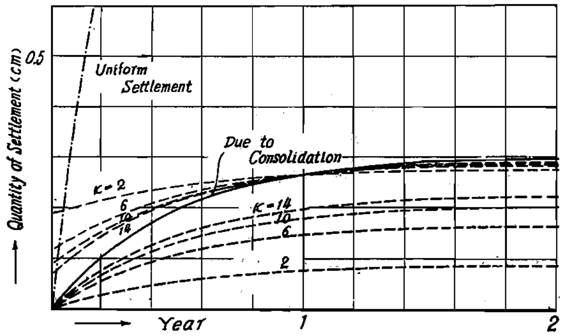

Fig. 16 The curves of differential settlement due to consolidation and elastic deformation both when κ varies. (Lower dashed curves: when the initial settlements are assumed zero, the dash-dotted curve and the real curve: uniform settlement and differential settlement due to consolidation alone, respectively)

It is clearly known from Fig. 15 that the curve of the function of differential settlement in the case which is caused by consolidation and elastic deformation of foundation both is not much different from that curve in the case which is caused by consolidation only. As for Fig. 16, the differential settlement at $N=0$ varies with the value of γ or κ and the settlement proceeds with this value as the initial value. The rate of differential settlement increases with the increasing of κ at the beginning, while final quantity of differential settlement due to consolidation and elastic deformation of foundation both has the tendency to become somewhat less than that of differential settlement due to consolidation only, but not much less.

Part B The method of calculation of final quantities of differential settlements of a structure

§ 10 Some considerations on final quantities of settlements

In Part A the behaviors of settlements in the case when underground clay stratum is consolidated have been discussed. In this connection, in order to discuss the characters of settlement process, some simplification could not be avoided concerning the derivation of differential equations of settlements, the assumption on load, the assumption of rigidity of a structure, and others. In fact, even as to symmetrical rigid frame, the calculation is pretty complicated in the case of more than 4 spans. However, if the consideration is confined to final quantities of settlements in negligence of settlement process, the following conclusions are gained from the above considerations and the treatment of equations become very much convenient. What matter in fact are final quantities of differential settlements.

1) When final quantities of settlements are considered, as for gradual increase of load, it is sufficient to consider the amount of final value of load. The variation of load on the way is not effective on final quantities of settlements.

2) When rigidity β gradually increases, differential settlements at some stage can exceed final quantities of settlements in the case of constant β as is shown in Chapter 9. In a numerical example, final quantity of settlement in the case of a 3-span symmetrical structure with constant β was exceeded by one tenth. But from the practical standpoint, in Part B this effect is disregarded.

3) As for the conditions of strata sandwiching clay stratum, final quantities of settlements are equal whether they are onesidedly permeable or both-sidedly permeable.

4) The approximate solutions by (2) assumed at the derivation of settlement equation coincide with the strict solutions by (1) when $N \to \infty$.

5) Even when there exist several clay strata and their thicknesses are not uniform, the treatment is simple.

Thus, below, the method of calculation for final quantities of settlements is derived.

§ 11 The derivation of equations of final quantities of settlements

It is assumed that there exist several underground clay strata, whose thickness and constants are not always uniform, and they extend infinitely in the direction perpendicular to the rigid frame so far discussed, so the treatment is two-dimentional.

Equations of elastic settlements are

$$y_{ei} = \sum_k \gamma_{ik} R_k(N) \,, \qquad \cdots\cdots\cdots(59)$$

where $R_k(N)$: reaction of base k which gradually increases with time as

$$R_k(N)_{N\to\infty} \rightarrow R_k \,,$$

γ_{ik} : effect of reaction of base i on elastic settlement of base k.

Equations of settlements due to consolidation, when pressure ${}_rq_i$ on clay stratum fluctuates by means of Terzaghi's formula of consolidation, are

$$y_{ci} = \sum_r {}_r\lambda_i \int_0^N {}_rq_i(\nu)\,{}_r\phi_i'(N-\nu)\,d\nu \,, \qquad \cdots\cdots\cdots(60)$$

where

$$ {}_r\phi_i(N) = 1 - \frac{8}{\pi^2} \sum_s \frac{1}{s^2} e^{-{}_r\mu_i s^2 N} \qquad s = 1,\ 3,\ 5, \cdots\cdots,$$

$$ {}_r\phi_i'(N) = \frac{d\,{}_r\phi_i(N)}{dN}, \qquad {}_r\mu_i = {}_rN_i/N \,,$$

$$ {}_rN_i = \frac{\kappa\pi^2\,{}_rC_i t}{{}_rd_i^2}, \qquad {}_r\lambda_i = {}_ra_{i\,r}d_i/(1 + {}_re_i) \,,$$

and ${}_rq_i(N)$: pressure on the clay stratum under i base

$$({}_rq_i)_{N\to\infty} = {}_rq_i,$$

κ : when permeable strata are on both sides of clay stratum, $\kappa = 1$,

when permeable stratum is on one side of clay stratum, $\kappa = \frac{1}{4}$.

(60) also are

$$\begin{aligned} y_{ci} &= \sum_r {}_r\lambda_i \int_0^N \frac{8}{\pi^2} \sum_s {}_r\mu_i\, {}_rq_i(\nu) e^{-{}_r\mu_i s^2 (N-\nu)} d\nu \\ &= \sum_s \sum_r {}_r\lambda_i \frac{8}{\pi^2}\, {}_r\mu_i e^{-{}_r\mu_i s^2 N} \int_0^N {}_rq_i(\nu) e^{{}_r\mu_i s^2 \nu} d\nu \\ &= \sum_s \sum_r {}_ry_{si}\,. \end{aligned} \qquad \cdots\cdots\cdots(61)$$

If the case of $N \to \infty$ is considered in (59) and (61), from (59)

$$y_{ei}=\sum_k \gamma_{ik}R_k, \qquad \cdots\cdots(62)$$

and from (61)

$$\frac{d_r y_{si}}{dN}=-{}_r\lambda_i\frac{8}{\pi^2}{}_r\mu_i{}^2 s^2 e^{-{}_r\mu_i s^2 N}\int_0^N {}_rq_i(\nu)e^{{}_r\mu_i s^2\nu}d\nu+{}_r\lambda_i\frac{8}{\pi^2}{}_r\mu_i\,{}_rq_i(N)$$
$$=-{}_r\mu_i s^2\,{}_ry_{si}+{}_r\lambda_i\frac{8}{\pi^2}{}_r\mu_i\,{}_rq_i(N)\ ,$$

then $N\to\infty$,

$${}_ry_{si}={}_r\lambda_i\frac{8}{\pi^2}\frac{1}{s^2}{}_rq_i \qquad \cdots\cdots(63)$$

Accordingly final quantity of settlement of i base is

$$y_i=y_{ei}+\sum_r\sum_s {}_ry_{si}$$
$$=\sum_k\gamma_{ik}R_k+\sum_r\sum_s {}_r\lambda_i\frac{8}{\pi^2}\frac{1}{s^2}{}_rq_i$$
$$=\sum_k\gamma_{ik}R_k+\sum_r {}_r\lambda_i\,{}_rq_i \qquad \cdots\cdots(64)$$

When base reactions in the case when settlements are not considered or bases are assumed to be completely fixed as in the ordinary design of structure are expressed as R_{i0},

$$R_i=R_{i0}+\sum_k\beta_{ik}y_k, \qquad \cdots\cdots(65)$$

$${}_rq_i=\sum_k {}_ra_{ik}R_k=\sum_k {}_ra_{ik}(R_{k0}+\sum_j\beta_{kj}y_j), \qquad \cdots\cdots(66)$$

and when (65) and (66) are substituted into (64),

$$y_i=\sum_k\gamma_{ik}(R_{k0}+\sum_j\beta_{kj}y_j)+\sum_k\sum_r {}_r\lambda_i\,{}_ra_{ik}(R_{k0}+\sum_j\beta_{kj}y_j)$$
$$=\sum_k(\gamma_{ik}+\sum_r {}_r\lambda_i\,{}_ra_{ik})R_{k0}+\sum_j\sum_k(\gamma_{ik}+\sum_r {}_r\lambda_i\,{}_ra_{ik})\beta_{kj}y_j. \qquad \cdots\cdots(67)$$

If

$$\left.\begin{aligned}\delta_{ik}&=\gamma_{ik}+\sum_r {}_r\lambda_i\,{}_ra_{ik},\\ a_{ij}&=\sum_k\delta_{ik}\beta_{kj},\\ K_{i0}&=\sum_k\delta_{ik}R_{k0},\end{aligned}\right\} \qquad \cdots\cdots(68)$$

then (67) become

$$y_i-\sum_j a_{ij}y_j=K_{i0}. \qquad \cdots\cdots(69)$$

(69) are the simultanious equations of final quantities of settlements by elasticity of foundation and consolidation of clay strata. Terms of uniform settlement and rotational settlement of a structure as a whole independent of

differential settlements are further eliminated from (69).

If $\bar{x}_i$ is set for the distance between base i and $x_0=\sum_i x_i/n$ (centre of gravity) when the same weights are considered to be on the points of action of K_{i0},

$$K_i^*=\bar{K}_0+c\bar{x}_i, \qquad \cdots\cdots(70)$$

where if

$$\sum_i K_i^*=\sum_i K_{i0}=\mathfrak{N} \qquad \cdots\cdots(71)$$

and

$$\sum_i K_i^*\bar{x}_i=\sum_i K_{i0}\bar{x}_i=\mathfrak{M} \qquad \cdots\cdots(72)$$

are assumed, from (71)

$$\mathfrak{N}=n\bar{K}_0+c\sum_i\bar{x}_i=n\bar{K}_0,$$

therefore

$$\bar{K}_0=\mathfrak{N}/n, \qquad \cdots\cdots(73)$$

and from (72)

$$\sum_i\bar{K}_0\bar{x}_i+\sum_i C\bar{x}_i^2=C\sum_i\bar{x}_i^2=CI_0=\mathfrak{M},$$

therefore

$$C=\mathfrak{M}/I_0, \qquad I_0=\sum_i\bar{x}_i^2. \qquad \cdots\cdots(74)$$

Settlement equations by K_i^* are expressed as follows:

$$y_i^*-\sum_j a_{ij}y_j^*=K_i^*. \qquad \cdots\cdots(75)$$

If $y_i^*=a+b\bar{x}_i$, then $a_{ij}y_j^*=0$.
Therefore

$$y_i^*=a+b\bar{x}_i=K_i^*=\bar{K}_0+c\bar{x}_i, \qquad \cdots\cdots(76)$$

hence the following equations are gained as equations of differential settlements,

$$y_i'-\sum_j a_{ij}y_j'=K_i', \qquad \cdots\cdots(77)$$

where

$$K_{i0}-K_i^*=K_i', \qquad y_i-y_i^*=y_i'.$$

In the above equations, it is not always necessary to assume line loads as to R_i and β as in Part A. The problem whether loads of columns in the direction perpendicular to the rigid frame so far discussed are taken as line loads or series of concentrated loads will be discussed in Chapter 13.

§ 12 On the method of solution of calculation equations of settlements

As is clarified from the above description, the uniform and rotational settlements obtained from (75) are indipendent of rigidity β and are determined by the total of R_i and the deviation of each base from centre of gravity. Equations of differential settlements (77) can be expressed as

$$(1-a_{ii})y_i' - \sum_j a_{ij}y_j' = K_i' \qquad (j \neq i). \qquad \cdots\cdots(78)$$

Therefore y_i' are linearly proportional to K_i' and K_j', accodingly to R_{i0} and R_{j0}. In this equation, coefficients $(1-a_{ii})$ are in general greater than those of other terms, so the solution by means of Iteration Method is possible.

§ 13 Coefficients

a) *as to* a

If Boussinesq's solution is taken to be available as in Part A and a for line load is adopted, (6) becomes

$$a = \frac{2h^3}{\pi r_{ji}^4} = 2/\pi h\{1+(x'l'/h)^2\}^2 = 2/\pi h\{1+x'^2\xi'^2\}^2, \qquad \cdots\cdots(79)$$

therefore

$$a'' = al' = 2\xi/\pi\{1+x'^2\xi'^2\}^2, \qquad \cdots\cdots(80)$$

where l': span length of transversal rigid frame (to be assumed as equal span length) in the perpendicular direction,

$$\xi' = l'/h, \qquad x = x'l'.$$

When loaded base plates which have constant dimentions are considered to be set in a line infinitely with equal span length in the perpendicular direction, as for the calculation of a'', from Newmark's formula[14],

$$\left.\begin{aligned}
&\Delta a'' = \Delta a l' = \frac{l'^2}{4B^2}I_\sigma = \frac{1}{4b^2}I_\sigma,\\
&I_\sigma = \frac{1}{4\pi}\left[\frac{2mn\sqrt{m^2+n^2+1}}{m^2+n^2+m^2n^2+1}\,\frac{m^2+n^2+2}{m^2+n^2+1}\right.\\
&\qquad\left. +\tan^{-1}\frac{2mn\sqrt{m^2+n^2+1}}{m^2+n^2+1-m^2n^2}\right],\\
&B = rl', \quad m = D/h, \quad n = L/h,
\end{aligned}\right\} \qquad \cdots\cdots(81)$$

therefore a'' is obtained by adding $\Delta a''$ of each loaded base plate by means of this equation (Fig. 17). The case of $r=0.5$ coincides with the case of strip load[15] which has the breadth $2\times 0.5l'$

$$\alpha''=\frac{1}{2\pi r}\left\{\sin 2\varepsilon\cos 2\psi+2\varepsilon\right\}$$
$$=\frac{1}{2\pi r}\left\{\frac{(x'+b)\xi'}{(x'+b)^2\xi'^2+1}-\frac{(x'-b)\xi'}{(x'-b)^2\xi'^2+1}\right.$$
$$\left.+\sin^{-1}\frac{(x'+b)\xi'}{\sqrt{(x'+b)^2\xi'^2+1}}-\sin^{-1}\frac{(x'-b)\xi'}{\sqrt{(x'-b)^2\xi'^2+1}}\right\}\cdots\cdots\cdots\cdots(82)$$

with $r=0.5$, simplifies the calculation (Fig. 18).

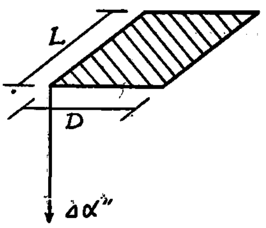

Fig. 17 The application of Newmark's equation to calculate α''.

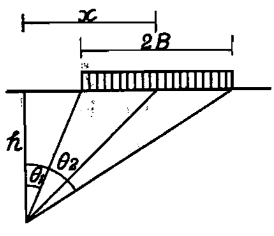

Fig. 18 The calculation of α'' caused by strip load.

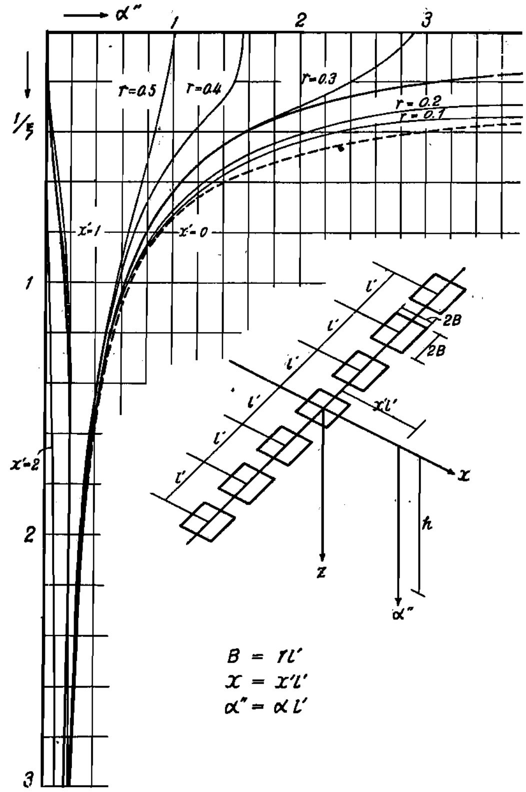

Fig. 19 The comparison among α'' caused by line load (: the thick real curves), infinite series of concentrated loads (:the dashed curves) and series of square loaded plates (:the fine real curves).

Equivalent line load (kg/cm) has been taken as unit 1 in (81) and (82) for the mutual comparison of α''. α'' for $x'=0$, 1, 2 in (80) through (82) are shown in Fig. 19. In the figure, the curves by infinite series of concentrated loads have been calculated by

$$a''=\frac{3\xi'^2}{2\pi}\left\{2\sum_{n=1}^{\infty}\frac{1}{\{(x'^2+n^2)\xi'^2+1\}^{5/2}}+\frac{1}{\{x'^2\xi'^2+1\}^{5/2}}\right\}. \qquad \cdots\cdots(83)$$

As is seen in this figure, a'' by a series of square loads differ only by less than one tenth from a'' by line load at $1/\xi=1.5$ or $h=1.5l'$ in the case of $x'=0$, and both rapidly approach to each other with the increasing of depth. Further the differences are very small at $x'=1$ and rapidly decrease with the increasing of x'

It is known from the above consideration that it is sufficient to use the equation for line load in general and to consider the dimentions of base plate when clay stratum is shallowly located.

b) *as to* γ

As to γ, Boussinesq solved the case of uniformly distributed load on a circular area on the surface of semi-infinite elastic body and Schleicher[16),17)] discussed the case of uniformly distributed load on a circular area and a rectangular one. According to them, in the case of distributed load on a circular area (radius is a), the ratio of settlement to equivalent line load 1 at S from centre of load is

$$\left.\begin{aligned}\gamma'&=\frac{4l'(1-\nu^2)}{\pi^2aE}F_2\left(\frac{S}{a},\frac{\pi}{2}\right) && S<a,\\ \gamma'&=\frac{4Sl'(1-\nu^2)}{\pi^2a^2E}\left[F_2\left(\frac{a}{S},\frac{\pi}{2}\right)-\left(1-\frac{a^2}{S^2}\right)F_1\left(\frac{a}{S},\frac{\pi}{2}\right)\right] && S>a,\end{aligned}\right\}\cdots\cdots(84)$$

where

$$F_2\left(k,\frac{\pi}{2}\right)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\varphi}\;d\varphi,$$

$$F_1\left(k,\frac{\pi}{2}\right)=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-k^2\sin^2\varphi}}d\varphi$$

In the case of loaded rigid plate,

$$\left.\begin{aligned}\gamma'&=\frac{l'}{2aE}(1-\nu^2) && S\leq a,\\ \gamma'&=\frac{l'}{\pi aE}(1-\nu^2)\sin^{-1}\frac{a}{S} && S\geq a.\end{aligned}\right\}\qquad\cdots\cdots(85)$$

The values of γ' by (84) and (85) are shown in Fig. 20. Therefore γ_{ij} are obtained as follows,

$$\left.\begin{aligned}\gamma_{ii}&=\gamma'_{s=0}+2\sum_{n=1}^{\infty}\gamma'_{s=nl'}\,,\\ \gamma_{ij}&=\gamma'_{s=\overline{ij}}+2\sum_{n=1}^{\infty}\gamma'_{s=\sqrt{\overline{ij}^2+(nl')^2}}\,.\end{aligned}\right\}\qquad\cdots\cdots\cdots\cdots\cdots\cdots(86)$$

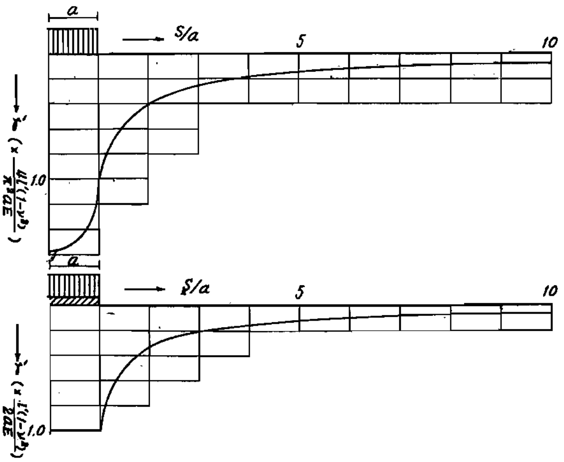

Fig. 20 γ' due to distributed load on a circular area (upper) and due to load on circular rigid plate (lower).

It is also shown according to Schleicher[16] that, value γ' due to distributed load on a square area does not differ much from that due to distributed load on a circular area. According to reference[16], γ', through the experiments on foundation, decreases more rapidly than in the above theory with the increasing of s.

c) as to β

As to β, it was determined in Part A with the assumption that the structure deforms in proportion to shearing force, but here it is treated more actually as follows.

It is assumed that rigid frame is composed of such elements as shown in Fig. 21 and rigidity of members is equal among beams and columns respectively in each story and span lengthes l are also equal. The derivation of values below will be shown in Appendix 2, here are only the results, where β is for the case of line load, and is $1/l'$ of rigidity of ordinary rigid frame.

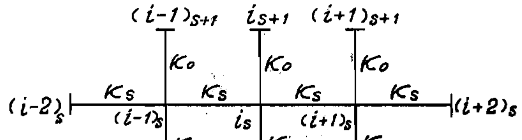

Fig. 21. Element of a rigid frame for calculation of β.

In the case of intermediate columns

$$\left.\begin{aligned}
\beta_{ii} &= -\sum_s\left(\frac{4}{3}-2\frac{\kappa_s}{f_s}\right)\kappa_s\beta_0\ ,\\
\beta_{i-1,i} &= \beta_{i+1,i}=\sum_s\frac{2}{3}\kappa_s\beta_0\ ,\\
\beta_{i-2,i} &= \beta_{i+2,i}=-\sum_s\frac{\kappa_s^2}{f_s}\beta_0\ ,\\
\beta_{ji}&(j\leq i-3 \text{ or } j\geq i+3)=0.
\end{aligned}\right\}\qquad\cdots\cdots\cdots\cdots\cdots\cdots(87)$$

In the case of columns at both ends

$$\left.\begin{aligned}
\beta_{11} &= \beta_{nn}=-\sum_s\kappa_s\left\{\frac{-\kappa_s(1-\kappa_s/f_s)^2}{f_{es}-\kappa_s^2/f_s}+\frac{2}{3}-\frac{\kappa_s}{f_s}\right\}\beta_0\ ,\\
\beta_{21} &= \beta_{n-1,n}=\sum_s\kappa_s\left\{\frac{-\kappa_s(1-\kappa_s/f_s)}{f_{es}-\kappa_s^2/f_s}+\frac{2}{3}\right\}\beta_0\ ,\\
\beta_{31} &= \beta_{n-2,n}=-\sum_s\frac{\kappa_s^2}{f_s}\frac{f_{es}-f_s}{f_{es}-\kappa_s^2/f_s}\beta_0\ ,\\
\beta_{i1} &= \beta_{jn}=0\quad(i\geq 4,\ j\leq n-3)\ ,\\
\beta_{12} &= \beta_{n,n-1}=\sum_s\kappa_s\left\{\frac{-\kappa_s^2(1/f_s-1/f_{es})}{f_s-\kappa_s^2(1/f_s+1/f_{es})}(1-\kappa_s/f_s)-\frac{\kappa_s}{f_{es}}+\frac{2}{3}\right\}\beta_0,\\
\beta_{22} &= \beta_{n-1,n-1}=-\sum_s\kappa_s\left\{\frac{-\kappa_s^3(1/f_s-1/f_{es})^2}{f_s-\kappa_s^2(1/f_s+1/f_{es})}-\kappa_s(1/f_{es}+1/f_s)+\frac{4}{3}\right\}\beta_0,\\
\beta_{32} &= \beta_{n-2,n-1}=\sum_s\kappa_s\left\{\frac{\kappa_s^2(1/f_s-1/f_{es})}{f_s-\kappa_s^2(1/f_s+1/f_{es})}+\frac{2}{3}\right\}\beta_0\\
\beta_{42} &= \beta_{n-3,n-1}=-\sum_s\frac{\kappa_s^2}{f_s}\frac{f_s-2\kappa_s^2/f_{es}}{f_s-\kappa_s^2(1/f_s+1/f_{es})}\beta_0\ ,\\
\beta_{i,2} &= \beta_{j,n-1}=0\qquad(i\geq 5,\quad j\leq n-4)
\end{aligned}\right\}\cdots(88)$$

where

$$\beta_0=18EK_0/l^2l',$$

K_0 : standard rigidity of member,

κ_s : rigidity ratio of beam of s-th story,

f_s : twice the total of rigidity ratios of members joined together at each intermediate joint of s-th story,

f_{es} : twice the total of rigidity ratios of members joined together at each joint of both ends of s-th story.

Thus the coefficients of a'', γ, and β are determined. The theoretical equations in Chapter 11 for line load can be expressed as follows.
When in (68)

$$\left.\begin{aligned}
\delta_{ik} &= \left(\frac{\gamma_{ik}l'}{\lambda_0}+\sum_r {}_r\mu_i\,{}_ra_{ik}''\right)\frac{\lambda_0}{l'}=\left(\zeta_{ik}+\sum_r {}_r\mu_i\,{}_ra_{ik}''\right)\frac{\lambda_0}{l'}=\bar{\delta}_{ik}\frac{\lambda_0}{l'},\\
a_{ij} &= \sum_k \bar{\delta}_{ik}\,\beta_{kj}\frac{\lambda_0}{l'}=\bar{a}_{ij}\frac{\lambda_0}{l'}\ ,\\
K_{i0} &= \sum_k \bar{\delta}_{ik}\,R_{k0}\frac{\lambda_0}{l'}=\bar{K}_{i0}\frac{\lambda_0}{l'}\ ,\\
{}_r\mu_i &= {}_r\lambda_i/\lambda_0, \qquad \zeta_{ik}=\gamma_{ik}l'/\lambda_0,
\end{aligned}\right\}\cdots\cdots\cdots(89)$$

and

$$\left.\begin{aligned}
&\bar{\mathfrak{N}}=\sum_i \bar{K}_{i0}, \qquad \overline{\mathfrak{M}}=\sum_i \bar{K}_{i0}\bar{x}_i\ ,\\
&\bar{\bar{K}}_0=\bar{\mathfrak{N}}/n, \qquad \bar{C}=\overline{\mathfrak{M}}/I_0\ ,\\
&\bar{K}_{i0}-\bar{K}_i{}^*=\bar{K}'_i\ ,
\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots(90)$$

then from (76) and (78)

$$y_i{}^*=(\bar{\bar{K}}_0+\bar{c}\bar{x}_i)\frac{\lambda_0}{l'} \qquad\cdots\cdots\cdots\cdots\cdots\cdots(91)$$

and

$$\left(\frac{l'}{\lambda_0}-\bar{a}_{ii}\right)y_i'-\sum_j \bar{a}_{ij}y_j'=\bar{K}_i' \quad (j\neq i) \qquad\cdots\cdots\cdots\cdots\cdots\cdots(92)$$

Then it is convenient to make such a table as shown in Fig. 22 for calculating coefficients and composing settlement equations on the basis of soil profil. In this figure, the case of 3-span structure over two clay strata is treated.

§ 14 Numerical example 2

Such a 4-story, 4-span symmetrical rigid frame (D rigid frame in the Instructions for Calculation of Reinforced Concrete Structures[12]), as shown in Fig. 23 is adopted as a numerical example for the method of calculation above. For the underground clay stratum $d=3$m and $h=10$ m are assumed, and for clay constants

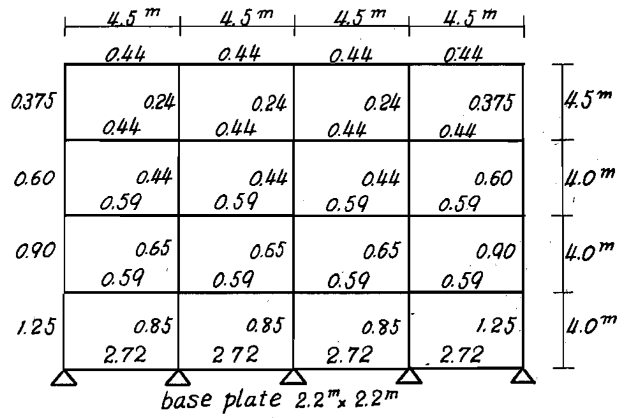

Fig. 23 The 4-span, 4-story symmetrical rigid frame adopted in Chapter 14. The value on each member expresses ratio of its rigidity.

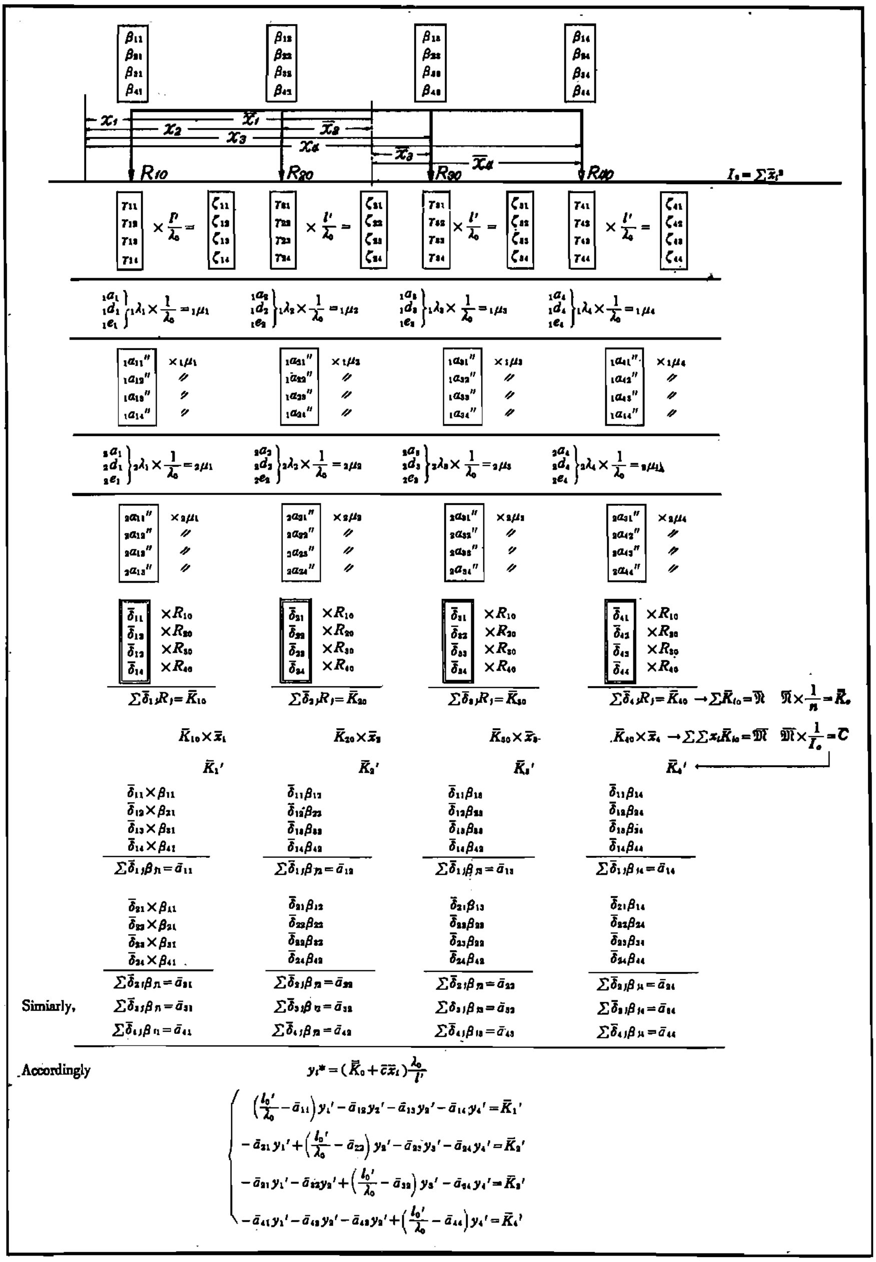

Fig. 22 Table for calculating coefficients and composings settlement equations on the basis of soil profil.

the same values as in (52) are taken. As to γ, (85) and (86) are adopted, and the effect of base reaction is assumed to work up to the neighbouring base and not farther. That is, $n=1$ is set in (86).

If β and ζ are calculated in the above example, the first and the second tables are obtained. Here

−107.7	+148.1	−65.6	0	0
+157.3	−315.0	+237.9	−64.1	0
−49.6	+231.0	−344.6	+231.0	−49.6
0	−64.1	+237.9	−315.0	+157.3
0	0	−65.6	+148.1	−107.7

Table 1 Table of β_{ij}.

1.32	0.40	0	0	0
0.40	1.32	0.40	0	0
0	0.40	1.32	0.40	0
0	0	0.40	1.32	0.40
0	0	0	0.40	1.32

$\times\dfrac{(1-\nu^2)l'^2}{2aE\lambda_0}$

Table 2 Table of ζ_{ij}.

0.3183	0.2201	0.0972	0.0400	0.0177
0.2201	0.3183	0.2201	0.0972	0.0400
0.0972	0.2201	0.3183	0.2201	0.0972
0.0400	0.0972	0.2201	0.3183	0.2201
0.0177	0.0400	0.0972	0.2201	0.3183

Table 3 Table of α_{ij}''.

$$K_0=2\times10^5\ \text{cm}^3,\qquad l=4.5\ \text{m},\qquad l'=5.0\ \text{m},\qquad \lambda_0=10\ \text{cm}^3/\text{kg},$$
$$R_0=R_1=R_5=132\ \text{kg/cm},\qquad R_2=R_3=R_4=1.1R_0,$$

where the self weight of the underground part of the base is not considered. At the calculation of rigidity, the whole sections of concrete of members are considered and $E=210{,}000\ \text{kg/cm}^2$ is set. Therefore $\beta_0=74.67\ \text{kg/cm}^2$. In the case of considering elastic settlement, $(1-\nu^2)l'^2/2Ea\lambda_0=0.05$ is set. This value is as much as in the case of dry fine sand.

a'' of the case in which a structure stands on the ground is as shown in the third table, and the result of calculation by (91) and (92) is as shown in Fig. 24. In this figure the settlements in the case caused by consolidation of clay stratum alone and in the case caused by both consolidation and elastic deformation of foundation are shown, and in its upper part the distributions of differential settlements are shown for comparison.

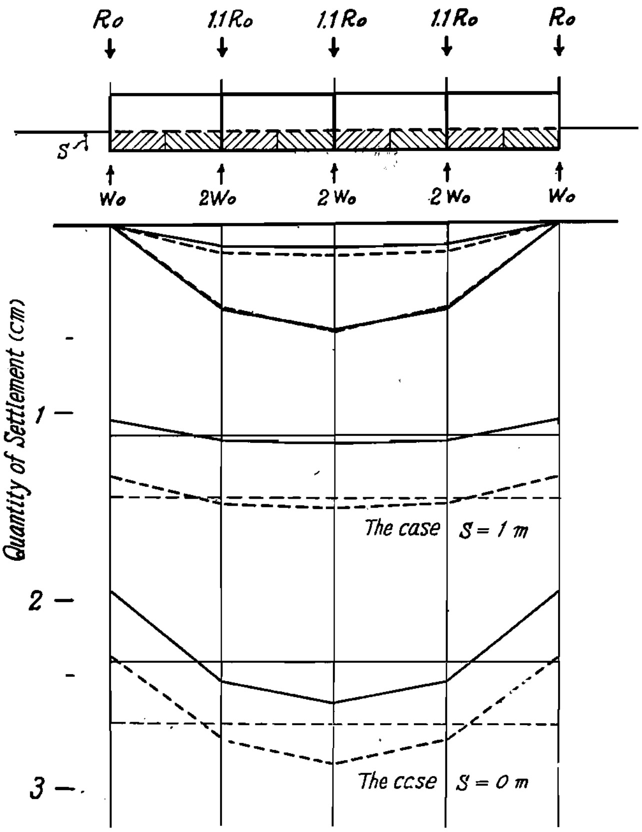

Fig. 24 Final quantities of settlements of bases in the case of a 4-span, 4-story rigid frame, where $d=3$m, $h=10$m when it stands on the surface and $h=9$m when it has semi-basement. The real lines represent the cases due to consolidation alone and the dashed lines represent the cases due to consolidation and elastic deformation of foundation.

In the case of a structure with basement or semi-basement, it is admitted by Terzaghi and Peck[18] too as a method of the reducing of differential settlements that, for settlement due to consolidation, the weight of soil driven out can be omitted from base reaction. In this connection, the whole base reactions must be considered for elastic settlements. In Fig. 24 are shown the final quantities of settlements in the case when the structure mentioned above is constructed being founded on the level as deep as 1 m from the surface. A structure with basement is more effective than a structure without it on differential settlements, and to have semi-basement or basement is considered to be very effective in the reducing of differential settlements.

As the result of two kinds of calculations above, the differential settlements in the case of a structure on the surface of foundation are the greatest at both end spans, and the moment distributions in members of structure caused by both elasticity of foundation and consolidation are calculated as in Fig. 25. The left half of the figure is for a structure on the surface of foundation and the right half is for a structure with the semi-basement. Therefore the maximum increments of fibre stresses to be caused in base beam and 1st floor beam are as followings.

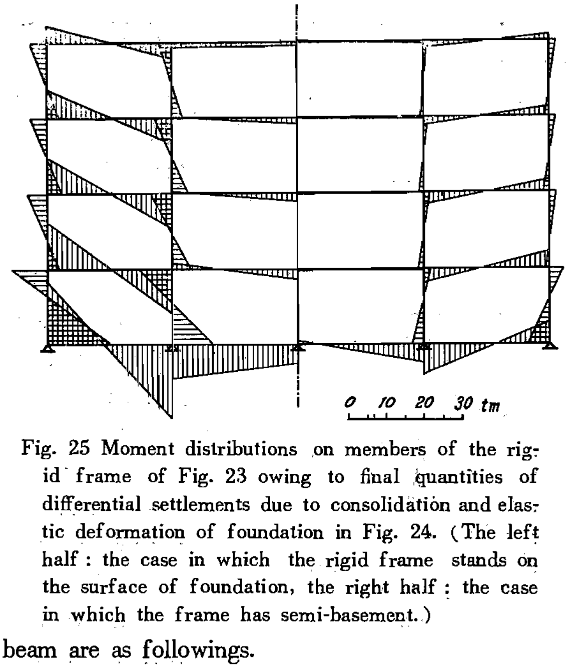

Fig. 25 Moment distributions on members of the rigid frame of Fig. 23 owing to final quantities of differential settlements due to consolidation and elastic deformation of foundation in Fig. 24. (The left half : the case in which the rigid frame stands on the surface of foundation, the right half : the case in which the frame has semi-basement.)

structure on the surface of foundation	structure with the semi-basement	
$y_{2\sim1} = 0.4408$ cm,	$y_{2\sim1} = 0.1463$ cm,	
$M_{0,max.} = 20.45$ ton·m,	$M_{0,max.} = 8.07$ ton·m,	
$M_{1,max.} = 12.65$ ton·m,	$M_{1,max.} = 4.26$ ton·m,	·········(93)
$\Delta\sigma_{0,max.} = 37.8$ kg/cm^2,	$\Delta\sigma_{0,max.} = 14.8$ kg/cm^2,	
$\Delta\sigma_{1,max.} = 51.7$ kg/cm^2,	$\Delta\sigma_{1,max.} = 17.4$ kg/cm^2.	

As in Chapter 9, $\Delta\sigma_0$ and $\Delta\sigma_1$ show the increments of fibre stresses to be caused in base beam and 1st floor beam respectively. But here the whole rectangular section of concrete is taken as beam section and the reinforcement are not considered. The damages caused by differential settlements on a structure can be explained to some extent from these numerals.

As a supplement, rigidity is somewhat greater in the definition of β in Chapter 4 than in its definition in Chapter 13, and the differential settlements become slightly smaller.

§ 15 Remarks on the application of the above method of calculation

A few remarks are to be added concerning the actual application of the method of calculation mentioned above on the basis of soil profile.

a) as to the thickness of stratum

For the sake of theorization, in order to calculate excess hydrostatic pressure on clay stratum, the applied value at the depth of centre of thickness of the stratum has been abopted and assumed to be constant through the thickness of the stratum. This assumption is available for thin stratum, but cannot helf result in inexactness with the inceasing of the thickness of the stratum. For higher approximacy, solutions are to be obtained by dividing the thickness of stratum into some strata, and by adopting the applied value of each depth of centre of thickness for each stratum with the assumption that there exist several clay strata. Because the condition of the direction of permeation is not effective on final quantities of settlements (cf. § 10).

b) as to a''

It was stated in Chapter 13 that a'' must be amended by considering breadths of rectangular base plates when the clay stratum is shallowly located, but definitions of a'' are all unified for equivalent line load, so (81)~(83) can be used as it is. This remark is to be applied for the definitions of β and γ as well.

c) Remarks on the numbers of transversal spans

The effect due to the difference between loads on the actual structure with finite transversal spans and the assumed line loads of infinite length in this essay is now to be discussed. In this connection, a'' by means of (81) and that by means of (83) are expressed as the sum of infinite series, whose convergency intends to become the function of depth, so the deeper the location of the clay stratum is, the more terms must be calculated. The numbers of terms are given in the Table 4 to obtain the value of more than 90 % of the convergent value gained through the calculation of Fig. 19. Thus when the structure has

h/l	0～0.6	0.8～1.5	2	3	5
n	0	1	2	3	7

Table 4

transversal spans whose number exceeds at least that numeral at each point of depth in the table, this method of calculation can be adopted by the exactness of more than 90 %. Here $n=0$ shows the case of only loaded plate just above.

Conclusion

As mentioned above, in Part A mainly the processes of settlements due to consolidation have been considered, and accordingly it has been clarified that the rigidity of a structure is greatly effective on differential settlements and does much work for reducing them. And the greater rigidity is, the greater the functions of differential settlement $(1-e^{-\Psi_i N})$ are in comparison with the function of uniform settlement $(1-e^{-N})$, so it is pointed out that the rates of differential settlements are promoted by rigidity. It has been considered that damages of structure intend to occur at the extention joints in the case of an extended structure, that the differences of thickness of underground clay stratum under bases intend to be greatly effective on differential settlements, and that damages of structure are likely to occur in the process of settlement in the case of such a structure as a reinforced concrete structure the rigidity of which gradually increases.

As for Part B, as the result of Part A to provide the enough rigidity for a structure against differential settlements become necessary, a method of calculation for final quantities of differential settlements has been shown. Further it is pointed out in a numerical calculation that differential settlements are considerably smaller in a case of a structure with basement or semi-basement than in the case of a structure on the surface of ground.

And it has been pointed out that in many cases base beams occupy the considerable part of the rigidity of structures and accordingly the rigidity of base beams is greatly effective on differential settlements.

It is natural that two-dimentional consolidation has influence when clay stratum is thick, although the assumption mentioned in the preface has not been related, and it will be necessary to examine the effect of the two-dimentional consolidation and the flow of clay, and that of creep deformation of a structure. However, these problems have to be left to the further study.

Acknowledgement

Since this study is a part of the research supported by the Grant in Aid for Miscellanious Scientific Research of the Ministry of Education, the co-authors would like to express our heartiest gratitude to the authority.

Appendix 1

By setting (1) approximately as (2), the simlification of the treatment of equations in Chapter 2 and the following chapters has been intended. In order to examine what effect this approximation results in, the solution by (1) is derived in the following and is compared with the solution by (2) by means of calculations.

In (1), when q is a function of N, from Duhamel's theorem,

$$y=2\lambda\sum_{m}e^{-M^2T}\int_0^T q(\tau)e^{M^2\tau}d\tau. \qquad \cdots\cdots\cdots\cdots(1')$$

If (8) is substituted into (1'),

$$y_i=2\lambda\sum_{m}e^{-M^2T}\int_0^T(K_i+\sum_{k}\delta_{ik}y_k)e^{M^2\tau}d\tau. \qquad \cdots\cdots\cdots\cdots(2')$$

If

$$y_{im}=2\lambda e^{-M^2T}\int_0^T(K_i+\sum_{k}\delta_{ik}\sum_{n}y_{kn})e^{M^2\tau}d\tau, \qquad \cdots\cdots\cdots\cdots(3')$$

then

$$\left.\begin{aligned}&\frac{dy_i}{dT}=\sum_{m}\frac{dy_{im}}{dT}=-\sum_{m}M^2y_{im}+2\lambda\sum_{m}e^{-M^2T}e^{M^2T}(K_i+\sum_{k}\delta_{ik}\sum_{n}y_{kn})\\&\qquad m=0,1,2,\cdots\cdots\infty\\&\qquad n=0,1,2,\cdots\cdots\infty.\end{aligned}\right\}\cdots(4')$$

Therefore the following equations are obtained for each value of m,

$$\frac{dy_i}{dT}+(M^2-2\lambda\delta_{ii})y_{im}-2\lambda K_i-2\lambda\sum_{k}\delta_{ik}\sum_{n}y_{kn}=0\ , \qquad \cdots\cdots\cdots\cdots(5')$$

where $(k=i,\ n=m)$ is assumed to be not satisfied simultaneously.

In order to solve (5'), it is assumed to be sufficient to consider as far as $m=n'$ according to the convergency of (1). If the form of solution is put as

$$y_{im}=\sum_{l} A_{iml}\,(1-e^{-\Psi_l T})\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(6')$$

which is substituted into (5′), then

$$-(M^2-2\lambda\delta_{ii}-\Psi_l)A_{iml}+2\lambda\sum_{k}\delta_{ik}\sum_{n}A_{knl}=0\ , \qquad \cdots\cdots\cdots\cdots\cdots\cdots(7')$$

$$(M^2-2\lambda\delta_{ii})\sum_{l}A_{iml}-2\lambda\sum_{k}\delta_{ik}\sum_{n}\sum_{l}A_{knl}-2\lambda K_i=0\ , \qquad \cdots\cdots\cdots\cdots\cdots\cdots(8')$$

$i,\ k$: number of columns $1,2,\cdots\cdots k'$,

$m,\ n$: $0,1,2,\cdots\cdots\cdots n'$.

Thus from determinant of coefficient terms of (7′) ratios of A_{iml} are obtained, from (8′) A_{iml} are determined, and solutions (6′) are obtained. Thus the solution by (1) can be obtained. However, the calculation by means of this method is greatly troublesome. The convergency of (1) is excellent except the vicinity of $T=0$. But, in order to satisfy the initial condition $y_{T=0}=0$, considerable number of terms are necessary. Therefore, in order to satisfy the initial condition and to give high approximacy to the curve of (1), the following equation is set.

$$y=\lambda q\{1-ce^{-N}-(1-c)e^{-5^2N}\}\ , \qquad \cdots\cdots\cdots\cdots\cdots\cdots(9')$$

where $\qquad c=8/\pi^2\ , \qquad N=\pi^2 T/4\ .$

The comparison of this curve with the curves of (1) and (2) is shown in Fig. 1′. In order to be compared with the case by the assumed curve of

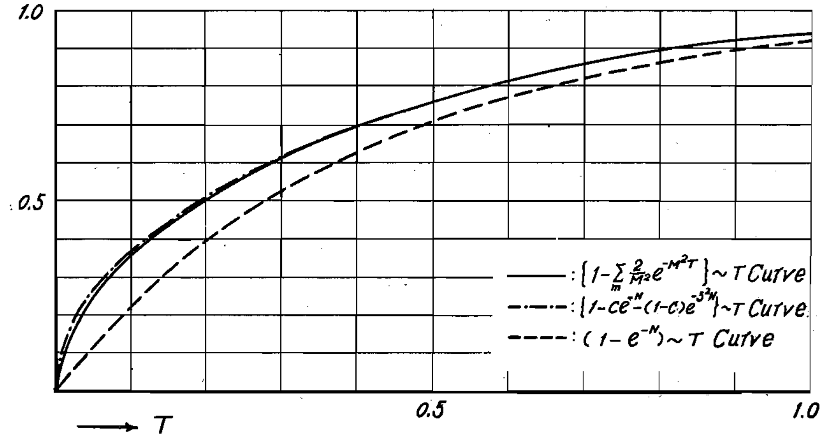

Fig. 1′ Comparison among functions of (2), (9′) and (11′). ($\xi=0.2$ and $d=10$ m)

(2), the case when the 3-span symmetrical rigid frame adopted in Chapter 9 and the same clay constant as in that chapter are used and $\xi=0.2$ and $d=10$m are set is considered.

In the case by (9′)

$$\left.\begin{aligned} y_1 &= 3.6619(1-e^{-N})+0.8558(1-e^{-26N})+0.4072(1-e^{-1.199N}) \\ &\quad +0.1222(1-e^{-26.23N}), \\ y_2 &= 3.6619(1-e^{-N})+0.8558(1-e^{-26N})+0.8235(1-e^{-1.199N}) \\ &\quad +0.2472(1-e^{-26.23N}), \end{aligned}\right\}\cdots\cdots\cdots(10')$$

and in the case by (2)

$$\left.\begin{aligned} y_1 &= 4.5159(1-e^{-N})+0.5294(1-e^{-1.258N}), \\ y_2 &= 4.5159(1-e^{-N})+1.0708(1-e^{-1.258N}), \end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots(11')$$

where $N=0.4678t'$, (t' : year).

From (10′) the first two terms become settlements of type A and the following two terms become settlements of type B, thus it is known that settlement of each base in this case is also considered to be divided into two parts. Moreover the functions of settlements type B are greater than that of uniform settlement. As for final quantities of settlements,

uniform settlements:

$$\left.\begin{aligned} &\text{from } (10') \quad 4.5177 \text{ cm}, \\ &\text{from } (11') \quad 4.5159 \text{ cm}, \end{aligned}\right\}\qquad\cdots\cdots\cdots\cdots\cdots\cdots(12')$$

differential settlements:

$$\left.\begin{aligned} &\text{from } (10') \quad 0.5413 \text{ cm}, \\ &\text{from } (11') \quad 0.5414 \text{ cm}, \end{aligned}\right\}\qquad\cdots\cdots\cdots\cdots\cdots\cdots(13')$$

and both coincide within the range of error. Then, in order to compare settlement curves, if they are shown by setting both final quantities of settlements type A and differential settlements 100 % respectively, Fig. 2′ is obtained. As known from this figure, the case of (10′) is somewhat faster than that of (11′) at initial N, and both cases intend to approach gradually to each other with the increasing of N.

As the result of the above consideration, with the assumption that (9′) is highly approximate to (1), if (1) is compared through (9′) with (2), both equations almost coincide in final quantities of settlements. But it is pointed out that the rates of differential settlements in the case caused by (2) intends

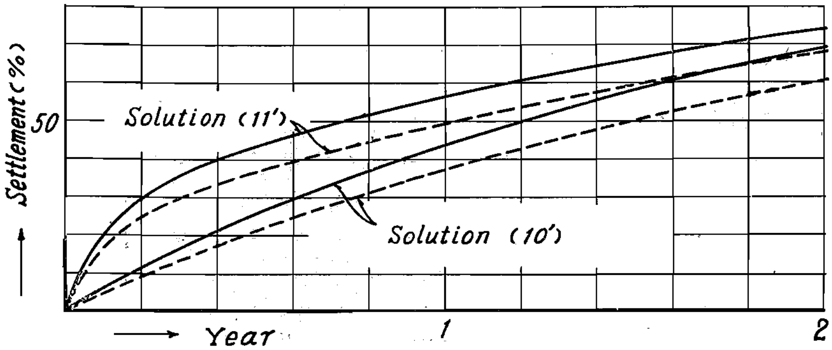

Fig. 2′ A-type and differential settlements due to consolidation in (10′) and (11′). (ξ=0.2 and d=10 m)

to be somewhat smaller than that in the case caused by (1) at initial N. However, to adopt (2) will be sufficient in the qualitative discussions of settlement process.

Appendix 2

As to the process of determining β values in Chapter 13 the case of intermediate columns is examined here.

If the standard degree of rigidity of member is set as K_0, in Fig. 21 rigidity of each member is expressed as follows.

$$k_s = K_0\kappa_s, \qquad k_0 = K_0\kappa_0 \text{ and } k_u = K_0\kappa_u. \qquad \cdots\cdots(14')$$

With the assumption that unit 1 of settlement is given at joint i, if the half of Fig. 21 is considered, moments at joints i and i-1 of s-th story are

$$\left.\begin{aligned} {}_sM_{i,i-1} &= 2EK_0\kappa_s(2\theta_i+\theta_{i-1}-3R) = (2\varphi_i+\varphi_{i-1}+\psi)\kappa_s, \\ {}_sM_{i-1,i} &= 2EK_0\kappa_s(2\theta_{i-1}+\theta_i-3R) = (2\varphi_{i-1}+\varphi_i+\psi)\kappa_s, \\ \text{here} \quad Rl &= -\psi l/6EK_0 = 1, \text{ then } \psi = -6EK_0/l. \end{aligned}\right\} \cdots\cdots(15')$$

From the equilibrium of moments at joint $(j-1)$

$$2(2\kappa_s+\kappa_0+\kappa_u)\varphi_{i-1}+\kappa_s\psi = 0,$$

therefore

$$\varphi_{i-1} = -\frac{\kappa_s}{f_s}\psi = \frac{\kappa_s}{f_s}\frac{6EK_0}{l}, \qquad \cdots\cdots(16')$$

where $\quad f_s$: twice the sum of ratios of rigidity of beams and columns joined together at joints s.

Therefore

$$\left.\begin{aligned} {}_sM_{i-1,i}&=\kappa_s(2\varphi_i+\psi)=2\kappa_s(\kappa_s+\kappa_0+\kappa_u)\psi/f_s,\\ {}_sM_{i,i-1}&=\kappa_s(\varphi_i+\psi)=2\kappa_s(1.5\kappa_s+\kappa_0+\kappa_u)\psi/f_s,\\ {}_sM_{i-1,i-2}&=2\kappa_s\varphi_i=-2\kappa_s^2\psi/f_s,\\ {}_sM_{i-2,i-1}&=\kappa_s\varphi_i=-\kappa_s^2\psi/f_s, \end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots(17')$$

accordingly in Fig. 3′

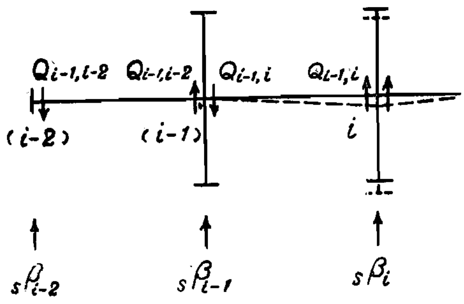

Fig. 3′

$$\left.\begin{aligned} {}_sQ_{i-1,i}&=-({}_sM_{i-1,i}+{}_sM_{i,i-1})/l=-\frac{2\kappa_s}{l}\left(1-\frac{3\kappa_s}{2f_s}\right)\psi,\\ {}_sQ_{i-1,i-2}&=-({}_sM_{i-1,i-2}+{}_sM_{i-2,i-1})/l=\frac{3\kappa_s^2}{lf_s}\psi. \end{aligned}\right\}\quad\cdots\cdots\cdots(18')$$

Then reactions at joints are

$$\left.\begin{aligned} {}_s\beta_{i-1}'&={}_sQ_{i-1,i}-{}_sQ_{i-1,i-2}=-\frac{2\kappa_s}{l}\psi\ ,\\ {}_s\beta_i'&=-2{}_sQ_{i-1,i}=\frac{4\kappa_s}{l}\left(1-\frac{3\kappa_s}{2f_s}\right)\psi\ ,\\ {}_s\beta_{i-2}'&={}_sQ_{i-1,i-2}=\frac{3\kappa_s^2}{lf_s}\psi \end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots(19')$$

Thus rigidity equation (87) for an unit of load is obtained by dividing (19′) by span-length l' in the perpendicular direction and by summing up about s. The same derivation as above is available in the case of columns in end spans.

Reference

1) K. Terzaghi and O. K. Frölich : Theorie der Setzung von Touschichten, 1939.
2) H. Meischeider : Setzungen von Ingenieurbauwerken, Der Bautechnik, Okt. 1954.
3) C. B. Crawford Settlement Studies on the National Museum Building, Ottawa, Canada, Proc. of the 3rd Intern. Conf. on Soil Mech. and Found. Eugineering, Session 4.
4) W. S. Hanna Settlements of Buildings on Preconsolidated Clay Layers, Proc. of the 3rd Intern. Conf. on Soil Mech. and Found. Engineering, Session 4.
5) K. Helenelund : Settlement Observations in Finland, Proc. of the 3rd Intern. Conf. on Soil Mech. and Found. Engineering, Session 4.
6) G. W. Holcomb, and P. P. Brown : Building Settlement Measurements- Oregon State College, Proc. of the 3rd Intern. Conf. on soil Mech. and Found. Engineering, Session 4.
7) A. W. Skempton, R. B. Peck and D. H. MacDonald : Settlement Analysis of six Structures in Chicago and London, Proc. of the Institute of Civil Engineer, Part I, vol. 4, No. 4, Jnly 1955.
8) Investigation Committee of Harbour Engineering of Osaka : A Study on the Settlement of the Ground in Osaka, June 1949. (in Japanese).
9) Y.Baba : On differential Settlements of Structures (Report l. Observation in Osaka and its Neighbourhood), Report of Architectural Institute of Japan, No. 24, Oct. 1953, (in Japanese).
10) Y.Koizumi and K. Takahashi : Observation of Differential Settlements of a Reinforced Concrete Structure and its Investigation, Reports of Architectural Institute of Japan, No.29, Oct. 1954 , (in Japanese).
11) M. A. Biot : Bending Settlement of a Slab Resting on a Consolidating Foundation, Jour. of Appl. Physics, Vol. 12, Feb. 1941.
12) Architectural Institute of Japan Instructions for Calculation of a Reinforced Concrete Structure and its Explanation, 1949, (in Japanese).
13) S. Ban Text-book on Reinforced Concrete Structures, 1948, (in Japanese).
14) K. Terzaghi Theoretical Soil Mechanics, 1954, p. 379 and 484.
15) O. K. Frölich Druckverteilung im Baugrunde, 1934, S. 39.
16) F. Schleicher Zur Theorie des Baugrundes, Der Bauingenieur, 1926, S. 931 und 949.
17) S. Ban : The Stresses and Settlements of Rigid Frames on Elastic Foundation and the Distribution of Base Reactions, Journal of Architectural Institute of Japan, Vol. 25, Apr. 1942, (in Japanese).
18) K. Terzaghi and R. B. Peck : Soil Mechanics in Engineering Practice, 1948, p. 448.

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Bulletin No. 14 Published March, 1956

昭和 31 年 3 月 25 日 印 刷
昭和 31 年 3 月 30 日 発 行

編輯兼発行者 京都大学防災研究所

印刷者 山代 多三郎
京都市上京区寺之内通小川西入

印刷所 山代印刷株式会社